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Monomial and Permutation Representation of Groups

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Rebeca Maria Blanquet

September 2012

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Abstract

This paper covers the construction of finite groups as homomorphic images of their progenitors. We will construct the group $(2 \times S_6) : 2^4$ as a homomorphic image of the progenitor $2^{*6} : S_6$, and we will prove by hand that $(2 \times S_6) : 2^4$ is isomorphic to $2^{*6} : S_6$ factored by two relations. We have a computer-based proof that the symmetric presentation

$$G = \left\langle x, y, t | x^6, y^2, (xy)^5, (yy^x)^3, \left(yy^{x^2}\right)^2, t^2, (xy, t), \left(t, y^{x^4}\right), [tt^y]^2, [xt]^5 \right\rangle$$

defines the group $(2 \times S_6) : 2^4$. With a by hand application of the double coset enumeration technique, we will prove that

$$G \cong (2 \times S_6) : 2^4$$

I will also describe the constuction of $A_5 \times A_5 : 2$, $A_7 \times S_5$, and A_7. as finite homomorphic images of progenitors $2^{*4} : S_4$, $3^{*5} :_m S_5$, and $3^{*5} :_m S_5$ respectively.

The main result of my thesis is the symmetric geberation of $M_{22} \times 2$, where M_{22} is the Mathieu sporadic group on 22 letters.

ACKNOWLEDGEMENTS

I would like to thank my advisor Dr. Zahid Hasan, for his patience and guidance these past few years. He has always been very helpful, even before he became my advisor and I am very grateful for this. I am very honored to have been able to work under his supervision. I also want to thank my two subadvisors, Dr. Min-Lin Lo and Dr. Laura Wallace, whom have provided me with excellent skills for my career goals. Thank you Dr. Wallace for your indispensable advice and feedback. Thank you Dr. Lo for helping me see all that I could accomplish by pushing myself every day to become a better student and mathematician. Thank you for giving me the opportunity to expand my knowledge, and gain experience as a research assistant and as a seasonal instructor. I would like to thank the Mathematics Department, for helping me through the M.A. program. I would like to thank my parents and siblings for their motivation and support throughout my college education. Jose, Rebeca, Jesusita, Francisco, and Manuel thank you for believing in me, this thesis is dedicated to all of you. I would also like to thank all my friends and family whom I have not mentioned. I am thankful for all of the support.

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Chapter 1

Introduction

When we wish to work with small finite groups, we use permutations and matrices. Although these two methods are the general methods of working with groups, they are not always effecient. In fact, when we wish to work with groups of finitely large orders, for example $M_{22} \times 3$ which has $887,040$ elements, these two methods are very inconvenient. While we have computer programs like MAGMA and GAP that can apply these two methods to such groups, these are still verydifficult to work with.

The main purpose of this thesis is to introduce another method of working with groups, that is more efficient when the groups we wish to work with are of a significantly large finite order. The method which we introduce in this chapter and that will be applied in the subsequent chapters is the called the manual double coset enumeration technique. The only restrictions on this technique is that it can only be performed on groups that are symmetrically generated by involutions. The manual double coset enumeration technique allows us to write elements in such large groups in a simpler and much more consice form.

1.1 Symmetric Generation of a Group

The *free product* of n copies of cyclic groups of order m is defined as
$E \cong m^{*n} =< t_0, t_1, ..., t_{n-1} | t_0^m = t_1^m = ... = t_{n-1}^m = 1 >$
So, E consists of all finite products of the involutory generators t_i without adjacent repetitions, i.e. $E =< t_{k_1} t_{k_2} ... t_{k_r} | r \in \mathbb{N}, k_i = k_{i+1} \text{ for } i = 1, 2, ..., r-1 >$.

If $m = 2$, then $t_1^2 = ... = t_{n-1}^2 = 1$ are called *symmetric generators.*
Here, $\mathbb{N}$ denotes the natural numbers, with zero, and r denotes the empty word, which is interpreted as the identity 1. Such finite products are called *reduced words* in the t_i.

Let G be a group. Let $T = \{t_0, t_1, ..., t_{n-1}\}$ be a subset of G such that $t_i^m = 1$. Then by defining $T_i = \langle t_i \rangle$ and $\bar{T} = \{T_0, ..., T_{n-1}\}$, we can define $N = N_{G(\bar{T})}$, the set normalizer of $\bar{T}$ in G. We say that T is a *symmetric generating set* for G if the following two conditions hold:
(1) $G = \langle T \rangle$
(2) N permutes $\bar{T}$ transitively.
In this case N is called the *control subgroup.* The conditions (1) and (2) imply that G is a homomorphic image of the progenitor $m^{*n} : N$, where m^{*n} is a free product of n copies of the cyclic groups of order m [Rot95]. Let $\pi \in N$, then we have

$$t_i^{\pi} = t_j^r \tag{1.1}$$

where r is an integer that is relatively prime to m. Notice that,since all the elements in N can be gathered on the left hand side, every element in the progenitor can be can be represented as πw, where $\pi \in N$ and w is a reduced word in the symmetric generators. For this representation to be unique, w must be simplified so that the adjacent symmetric generators are distinct. Hence, to obtain G, we can factor the progenitor by any additional relator that is of the form

$$\pi w\,(t_1, t_2, ..., t_n) \tag{1.2}$$

where $\pi \in N$ and w is a reduced word in T.

By factoring the progenitor $m^{*n} : N$ by such relators, we obtain a *factor group.* In the preceding chapters, the groups we will be working with are presented as factor groups. Next, we must explain how these factor groups

$$\frac{m^{*n} : N}{\pi_1 w_1, \pi_2 w_2, \cdots} \tag{1.3}$$

can be identified through the manual double coset enumeration technique:

1.2 Manual Double Coset Enumeration Technique

We wish to obtain homomorphic image of the progenitor

$$m^{*n} : N \tag{1.4}$$

In this thesis we will focus on two cases, the case where $m = 2$ and the case where $m = 3$. In the case where $m = 2$, we seek homomorphic images of the progenitor $2^{*n} : N$ where N is a transitive permutation group on n letters. We begin by identifying the n free generators and N with their respective images.

$$\frac{2^{*n} : N}{\pi_1 w_1, ..., \pi_s w_s} \tag{1.5}$$

where $\pi \in N$, $t_i \in T$. Allow i to represent the symmmetric generator t_i in expressions such as the relation shown above [Cur07]. Since $t_i\pi = \pi t_{\pi(i)}$, or $i\pi = \pi i^{\pi}$, the permutations can be grouped on the left. Thus, any element g of G can be written as a permutation, π, in N followed by a word in the symmetric generators. That is, $g = \pi w$, where $\pi \in N$, w is a word in the symmetric generators. Suppose that NgN is a double coset of N in G. Then we have:

$$NgN = N\pi wN = NwN \tag{1.6}$$

Notice that since $\pi \in N$, it gets absorbed, hence $N\pi = N$ above. We denote the double coset NwN by $[w]$.

Example 1.1. *We use* [123] *to denote the double coset* $Nt_1t_2t_3N$.

The case where w is a word of length zero, namely the identity element e, we have:

$$NwN = NeN = N \tag{1.7}$$

This double coset is denoted by $[*]$. We continue by defining the point stabilisers in N. The *single point stabiliser* in N is the the set

$$N^i = C_N(t_i) = \{n \in N | ni = in\} \tag{1.8}$$

The *two point stabiliser* in N is the the set

$$N^{ij} = C_N(\langle t_i \rangle), \tag{1.9}$$

and so on. The *coset stabilizing subgroup*, N^w of N is given by

$$N^w = \{\pi \in N : Nw\pi = Nw\}, \tag{1.10}$$

where w is a word in the symmetric generators. Notice that $N^w \leq N^{(w)}$. To obtain the number of single cosets in each double coset NwN, denoted $[w]$, we must divide

$$\frac{|N|}{|N^{(w)}|}$$

In order to obtain the index of N in G, we must perform a manual double coset enumeration of G over N. This process consists of finding all the double cosets, $[w]$ and calculating the number of single cosets contained in each $[w]$. The double coset enumeration is complete when the set of right cosets is closed under right coset multiplication. The Completion Test, the process of finding the point stabiliser, coset stabiliser, and number of single coset, is best performed by calculationg the orbits of $N^{(w)}$ on the symmetric generators. For each double coset $[w]$, take one symmetric generator, t_i, from each orbit and identify to which double coset Nwt_i belongs to. We will also find the double coset decomposition of G into double cosets NgN, where $g \in m^{*n} : N$ and find a set $\{g_1, g_2, ...\}$ of elements of G such that

$$G = Ng_1N \cup Ng_2N \cup ... \tag{1.11}$$

However, we previously found that $g_i = \pi_i w_i$, hence

$$G = N \cup Nw_2N \cup Nw_3N \cup ... \tag{1.12}$$

where w_1 denotes the identity, or words of lenght zero.

Chapter 2

Preliminary Information

2.1 Group

A *group* is a set, G, combined with an operation $*$, such that:
(1) The identity element exist:
There exists an $e \in G$ such that for all $g \in G$, $e * g = g * e = g$
(2) The inverse element exists in G:
There exists a $h \in G$ such that $g * h = h * g = e$
We say H is a *subset* of G if every element of H is also an element of G.

2.2 Semi-Direct Product

Lemma 2.1. *Let K be a group and $A \leq AutK$ be a subgroup of the automorphism group of K. Then the cartesian product $A \times K$ becomes a group under the binary operation " $\circ$ " defined by $(a, x) \circ (b, y) = (ab, x^b y)$ where $a, b \in A$ and $x, y \in K$.*

The group constructed from a cartesian product of two groups A and K, as described in the lemma above, is called a *semi-direct product* and is denoted by $K : A$.
A *progenitor* is a semi-direct product of the form:
$P \cong m^{*n} : N = \{\pi\omega | \pi \in N$ and ω is a reduced word in $t_i\}$
where m^{*n} denotes the free product of n copies of the cyclic group of order m generated by involutions t_i for $i = 1, 2, ..., n$; and N is a transistive permutation group of degree n which acts on the free product by permuting the involutory generators.

2.3 Group Action

Let G be a group and X be a nonempty set. We say that G acts on X if there exists a mapping $\alpha : G \times X \to X$ defined as $(g, x) \to gx$ such that:
(1) $1 \cdot x = x, \forall x \in X$
(2) $(gh)x = g(hx), \forall g, h \in X$
α is called an *action of G on X*.
The actions of the point stabiliser are by conjugation. If G is a group and $a \in G$, then a *conjugate* of a is any element in G of the form gag^{-1}, where $g \in G$. We also write $gag^{-1} = a^g$.

2.4 Single Coset

Let G be a group and H be a subgroup of G then a *single coset* is a set $Ha = \{ha | a \in G\}$. Notice that cosets partitionthe set G. Two properties of single cosets: (1) $Ha = Hb$ or $Ha \cap Hb = \oslash$
(2) $Ha = H \iff a \in H$.

2.5 Double Coset

Let H be a subgroup of G. Let $x \in G$. Then $HxH = \{Hxh | h \in H\}$ is the *double coset* of H in G. Notice that double cosets are composed of single cosets. The *index* of a subgroup H in G, denoted by $[G : H] = \frac{|G|}{|H|}$, is the number of single cosets of H in G. In particular, the number of single cosets in the double coset NwN is $\frac{|N|}{|N^{(w)}|}$. To determine the *single cosets* in a double coset NwN, you take Nw and conjugate it by its coset stabiliser $N^{(w)}$. If $N^{(w)}$ has several elements, you do this for each element. The *orbits* of $N^{(w)}$ on the symmetric generators are obtained through conjugation of each generator by $N^{(w)}$. The orbits are disjoint. If H is a subgroup of G and $a \in G$, then the *right coset* Ha is the subset Ha of G, where $Ha = \{ha : h \in H\}$.

2.6 Permutation Group

In some of the following chapters, we will be dealing with groups in which the control subgroup ,N, is a permutation group . The *Permutation Group* S_n is the group of permutations of (01234....n). The order of S_n is $|S_n| = n!$

Let $X = \{1, 2, 3, ...\}$. Then S_X, the set of all one-to-one and onto mappings from X to X, called permutations of X, forms a group under function composition. S_X is called the *permutation group* of X. If $X = \{1, 2, 3, ..., n\}$, then $S_X = S_n$ is called the *symmetric group of degree n.*

Chapter 3

The Construction of $(2 \times S_6) : 2^4$

We wish to prove that the symmetric representation of $(2 \times S_6) : 2^4$ is given by $\frac{2^{*6}:S_6}{[t_0t_1]^2;[(012345)t_0]^6}$, where $X \sim (012345)$ and $Y \sim (01)$. The symmetric presentation of the progenitor $2^{*6} : S_6$ is:
$2^{*6} : S_6 \cong \left\langle x, y, t | x^6 = y^2 = (xy)^5 = (yy^x)^3 = (yy^{x^2})^2 = 1 = t^2 = (xy, t) = (t, y^{x^4}) \right\rangle$.
Notice that the control subgroup, N, is S_6, the permutation group on six elements.

In order to prove:
$(2 \times S_6) : 2^4 \cong \left\langle x, y, t | x^6 = y^2 = (xy)^5 = (yy^x)^3 = (yy^{x^2})^2 = 1 = t^2 = (xy, t) = (t, y^{x^4}) \right\rangle$,
we must perform a double coset enumeration of $(2 \times S_6) : 2^4$ over S_6. The double coset enumeration consists of finding all double cosets $[w]$ and figuring out how many single cosets each of these contains. We know that the double coset enumeration is complete when the set of right cosets is closed under right multiplication. That is, when none of the t_i's can expand the cayley graph any further. From the definition of 2^{*6} we obtain the property $t_0^2 = t_1^2 = t_2^2 = t_3^2 = t_4^2 = t_5^2 = 1$. The control subgroup N, is permutation group S_6 which consists of the permutations of $0, 1, 2, 3, 4$ and 5. That is $S_6 = \langle (01), (012345) \rangle$. The number of elements contained in S_6, the *order* of S_6, is calculated as follows: $|S_6| = 6! = 720$

We will now proceed to expand our relators and obtain the relations which we can use to simplify our computations.

The first relation is as follows: $[t_0t_1]^2 = e$.
If we we expand the left hand side, we obtain the following: $t_0t_1t_0t_1 = e$
if we use the operation of right multiplication, we can further simplify this relation:

$t_0t_1t_0t_1 \cdot t_1 = t_1$

$\Rightarrow t_0t_1t_0t_1^2 = t_1$

$\Rightarrow t_0t_1t_0 = t_1$

$\Rightarrow t_0t_1t_0 \cdot t_0 = t_1 \cdot t_0$

$\Rightarrow t_0t_1t_0^2 = t_1 \cdot t_0$

$\Rightarrow t_0t_1 = t_1t_0$

Thus, we obtain the relation $t_0t_1 = t_1t_0$

The second relation is $[(012345)\, t_0]^6 = e$.

Let $\pi = (012345)$. Then, by permutation multiplication, we obtain the following:

$\pi^1 = (012345)$

$\pi^2 = (024)\,(135)$

$\pi^3 = (03)\,(14)\,(25)$

$\pi^4 = (042)\,(153)$

$\pi^5 = (054321)$

$\pi^6 = e$

So, if we expand $[(012345)\, t_0]^6 = e$ we obtain the following:

$\pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$

$\Rightarrow \pi\pi^5\pi^{-5}t_0\pi\pi^4\pi^{-4}t_0\pi\pi^3\pi^{-3}t_0\pi\pi^2\pi^{-2}t_0\pi\pi\pi^{-1}t_0\pi t_0 = e$

$\Rightarrow \pi^6\left(\pi^{-5}t_0\pi^5\right)\left(\pi^{-4}t_0\pi^4\right)\left(\pi^{-3}t_0\pi^3\right)\left(\pi^{-2}t_0\pi^2\right)\left(\pi^{-1}t_0\pi\right)t_0 = e$

$\Rightarrow \pi^6 t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi^1} t_0 = e$

$\Rightarrow (e)\, t_0^{(054321)} t_0^{(042)(153)} t_0^{(03)(14)(25)} t_0^{(024)(135)} t_0^{(012345)} t_0 = e$

$\Rightarrow t_5t_4t_3t_2t_1t_0 = e$

if we use the operation of right multiplication, we can further simplify:

$\Rightarrow t_5t_4t_3t_2t_1t_0^2 = t_0$

$\Rightarrow t_5t_4t_3t_2t_1^2 = t_0t_1$

$\Rightarrow t_5t_4t_3t_2^2 = t_0t_1t_2$

$\Rightarrow t_5t_4t_3 = t_0t_1t_2$

Thus, we obtain the relation: $t_5t_4t_3 = t_0t_1t_2$

The third relation is $t_5t_4t_3 = t_0t_1t_2$.

if we use the operation of right multiplication, we can further simplify this relation:

$\Rightarrow t_5 t_4 t_3^2 = t_0 t_1 t_2 t_3$

$\Rightarrow t_5 t_4 = t_0 t_1 t_2 t_3$

$\Rightarrow t_5 t_4^2 = t_0 t_1 t_2 t_3 t_4$

$\Rightarrow t_5 = t_0 t_1 t_2 t_3 t_4$

$\Rightarrow t_5 t_3 = t_0 t_1 t_2 t_3 t_4 t_3$

$\Rightarrow t_5 t_3 t_4 = t_0 t_1 t_2 t_3 t_4 t_3 t_4$

$\Rightarrow t_5 t_3 t_4 = t_0 t_1 t_2 \left(t_3 t_4\right)^2$

$\Rightarrow t_5 t_3 t_4 = t_0 t_1 t_2$

Thus, we obtain the following: $t_5 t_3 t_4 = t_0 t_1 t_2$

3.1 The Double Coset Enumeration of G over S_6

First, note that $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$. We will denote this double coset by $[*]$. The number of elements in $[*]$ is $\frac{|N|}{|N|} = \frac{720}{720} = 1$, hence $[*]$ consists of the single coset, N. Next, look at the double coset Nt_0N which we will denote by [0]. To obtain the elements in [0], we must first find the point stabiliser of 0, N^0. The point stabiliser consists of permutations in $N = S_6$ which fix 0 and permute $1, 2, 3, 4, 5$. This means that $\left|N^0\right| = 5! = 120$. In this case, we have that the set stabiliser, $N^{(0)} = N^0$. Thus, $\left|N^{(0)}\right| = 5! = 120$. To find the orbits of $N^{(0)}$ in $\{0, 1, 2, 3, 4, 5\}$ we conjugate $0, 1, 2, 3, 4$ and 5 by $N^{(0)}$. The orbits of $N^{(0)}$ are $\{0\}$ and $\{1, 2, 3, 4, 5\}$. The number of single cosets in the double coset [0] is obtained by dividing $\frac{|N|}{\left|N^{(0)}\right|} = \frac{6!}{5!} = 6$. To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(0)}$ in N. The transversals of $N^{(0)}$ in N are $Id(N), (0,1), (0,2), (0,3), (0,4)$ and $(0,5)$. Now we can apply the transversals to Nt_0 by conjugation to obtain the single cosets in [0]:

$N\left(t_0\right)^{Id(N)} = Nt_0$

$N\left(t_0\right)^{(0,1)} = Nt_1$

$N\left(t_0\right)^{(0,2)} = Nt_2$

$N\left(t_0\right)^{(0,3)} = Nt_3$

$N\left(t_0\right)^{(0,4)} = Nt_4$

$N\left(t_0\right)^{(0,5)} = Nt_5.$

Therefore $[0] = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, Nt_5\}$.

Look at the double coset $[*]$ consisting of the single coset N. There are six things that we can apply to N to obtain things in $[0]$, thus, six things expand the Cayley graph from $[*]$ to $[0]$. Next we must take a representative from each orbit of $N^{(0)}$ and apply it to Nt_0, by conjugation, to determine if it will extend or collapse the Cayley graph:

$Nt_0 \cdot t_0 = Nt_0^2 = N \in [*]$.
Since the orbit of 0 is of length 1,
one t_i will collapse the Cayley graph from $[0]$ to $[*]$.

$Nt_0 \cdot t_1 = Nt_0t_1$
This is a new double coset which we will label as $[01]$.
Since the orbit of 1 is of length 5,
five t_i will extend the Cayley graph from $[0]$ to $[01]$.

Now, we look at the double coset Nt_0t_1N which we denoted $[01]$. To obtain the elements in $[01]$, we must first find the point stabiliser of 0 and 1, N^{01}. The point stabiliser consists of permutations in $N = S_6$ which fix 0 and 1 and permute $2, 3, 4, 5$. This means that $\left|N^{01}\right| = 4! = 24$, so $N^{01} \cong S_4 = \langle (2,3), (2,3,4,5) \rangle$. The set stabiliser, $N^{(01)} \geq N^0$. Due to the relations mentioned in the previous section, $N^{(01)}$ increases:

From our relation, we have $t_0t_1 = t_1t_0$
$\Rightarrow Nt_0t_1\,(01) = N\,(t_0t_1)^{(01)} = Nt_1t_0 = Nt_0t_1$
$\Rightarrow (01) \in N^{(01)}$
So $N^{(01)} = \left\langle N^{01}, (01) \right\rangle$.

Thus, $\left|N^{(01)}\right| = 24 \times 2 = 48$. To find the orbits of $N^{(01)}$, we conjugate $0, 1, 2, 3, 4$ and 5 by $N^{(01)}$. The orbits of $N^{(01)}$ are $\{0\}, \{1\}$ and $\{2, 3, 4, 5\}$. The number of single cosets in the double coset $[01]$ is obtained by dividing $\frac{|N|}{\left|N^{(01)}\right|} = \frac{720}{48} = 15$. To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(01)}$ in N. The transversals of $N^{(01)}$ in N are: $Id(N), (1,2,3,4,5,0), (1,3,5)\,(2,4,0)$, $(1,2,3,4,5), (1,4)\,(2,5)\,(3,0), (1,3,5,2,4,0), (1,5,3)\,(2,0,4), (1,4)\,(2,5,3,0)$, $(1,3,5,2,4), (1,0,5,4,3,2), (1,5,2,4)\,(3,0), (2,5,3,0,4), (2,0,5,4,3), (1,0,4,2,5,3)$, and $(1,2,4)\,(5,0)$. Now, conjugate Nt_0t_1 by the set stabilizer $N^{(01)}$ to obtain a relation:

$Nt_0t_1^{N^{(01)}} = Nt_1t_0$ and our relation results in $01 \sim 10$.
Applying the transversals to this relation by conjugation yields the 15 single cosets in the double coset [01]:

$$(01 \sim 10)^{Id(N)} = 01 \sim 10$$
$$(01 \sim 10)^{(1,2,3,4,5,0)} = 12 \sim 21$$
$$(01 \sim 10)^{(1,3,5)(2,4,0)} = 23 \sim 32$$
$$(01 \sim 10)^{(1,2,3,4,5)} = 02 \sim 20$$
$$(01 \sim 10)^{(1,4)(2,5)(3,0)} = 34 \sim 43$$
$$(01 \sim 10)^{(1,3,5,2,4,0)} = 13 \sim 31$$
$$(01 \sim 10)^{(1,5,3)(2,0,4)} = 45 \sim 54$$
$$(01 \sim 10)^{(1,4)(2,5,3,0)} = 24 \sim 42$$
$$(01 \sim 10)^{(1,3,5,2,4)} = 03 \sim 30$$
$$(01 \sim 10)^{(1,0,5,4,3,2)} = 50 \sim 05$$
$$(01 \sim 10)^{(1,5,2,4)(3,0)} = 35 \sim 53$$
$$(01 \sim 10)^{(2,5,3,0,4)} = 41 \sim 14$$
$$(01 \sim 10)^{(2,0,5,4,3)} = 51 \sim 15$$
$$(01 \sim 10)^{(1,0,4,2,5,3)} = 40 \sim 04$$
$$(01 \sim 10)^{(1,2,4)(5,0)} = 52 \sim 25$$

Therefore, the fifteen distinct single cosets in this double coset are as follows:

$$[01] = \{Nt_0t_1, Nt_0t_2, Nt_0t_3, Nt_1t_2, Nt_1t_3, Nt_2t_3, Nt_2t_4, Nt_3t_4,$$
$$Nt_3t_5, Nt_4t_0, Nt_4t_1, Nt_4t_5, Nt_5t_0, Nt_5t_1, Nt_5t_2\}$$

Now we must take one representative from each orbit of $N^{(01)}$, so 0,1, and 2, apply each to Nt_0t_1 by right multiplication, and determine if these representatives will extend or collapse the Cayley graph.

$$Nt_0t_1 \underset{Relation1}{=} Nt_1t_0$$
$$\Rightarrow Nt_0t_1 \cdot t_0 = Nt_1t_0 \cdot t_0$$
$$\Rightarrow Nt_0t_1t_0 = Nt_1t_0^2 = Nt_1 \in [0].$$

Since the orbit of 0 is of length 1,

one t_i will collapse the Cayley graph from [01] to [0] .

$Nt_0t_1 \cdot t_1 = Nt_0 (t_1)^2 = Nt_0 \in [0]$.
Since the orbit of 1 is of length 1,
one t_i will collapse the Cayley graph from [01] to [0].

$Nt_0t_1 \cdot t_2 = Nt_0t_1t_2$.
This is a new double coset which we will label as [012].
Since the orbit of 2 is of length 4,
four t_i's will extend the Cayley graph from [01] to [012].

Thus, we have two things collapsing the Cayley graph from [01] to [0] and four things extending the Cayley graph from [01] to [012].

Now, we look at the double coset $Nt_0t_1t_2N$ which we denoted [012]. To obtain the elements in [012], we must first find the point stabiliser of $0, 1$, and 2, N^{012}. The point stabiliser consists of permutations in $N = S_6$ which fix $0, 1$ and 2 and permute $3, 4, 5$. This means that $\left|N^{012}\right| = 3! = 6$. So, $N^{012} \cong S_3 = \langle (3,4), (3,4,5) \rangle$. The set stabiliser, $N^{(012)} \geq N^{012}$. Now we must check if there are any additional permutations in S_6 , belonging to the set stabiliser. We do this by conjugation:

$$N\left(t_0t_1t_2\right)^{(5,0)(1,4)(2,3)} = Nt_5t_4t_3 \underset{Relation2}{=} Nt_0t_1t_2 \Rightarrow (5,0)(1,4)(2,3) \in N^{(012)}$$
$$N\left(t_0t_1t_2\right)^{(5,1,3,2,4,0)} = Nt_5t_3t_4 \underset{Relation3}{=} Nt_0t_1t_2 \Rightarrow (5,1,3,2,4,0) \in N^{(012)}$$
So, $\left\langle N^{012}, (5,0)(1,4)(2,3), (5,1,3,2,4,0) \right\rangle \leq N^{(012)}$

Therefore, $\left|N^{(012)}\right| = 6 \times 2 \times 6 = 72$. Note that we obtain the orbits of $N^{(012)}$ through the conjugation of $0, 1, 2, 3, 4$ and 5 by $N^{(012)}$. Since our first conjugation is $0^{N^{(012)}} = \{0, 5, 1, 3, 2, 4\}$, and yields all six numbers then there is no need to conjugate any of the remaining five numbers. Thus, the orbit of $N^{(012)}$ is $\{0, 5, 1, 3, 2, 4\}$. The number of single cosets in the double coset [012] is obtained by dividing $\frac{|N|}{\left|N^{(012)}\right|} = \frac{720}{72} = 10$. To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(012)}$ in N. The transversals are as follows:
$Id(N), (5,0), (1,5,4,2)(3,0),$
$(1,2,3,4,5,0), (1,0,5,4,3,2), (1,2,3,4,5), (1,3,5,2,4,0), (1,0,4,3,2), (1,5,4,2,0,3),$

and $(1,4)(2,5,3,0)$. Conjugating $Nt_0t_1t_2$ by the set stabiliser $N^{(012)}$, we obtain the following relations:

$N(t_0t_1t_2)^{(5,0)(1,4)(2,3)} = Nt_5t_4t_3$ and our relation results in $012 \sim 543$
$N(t_0t_1t_2)^{(5,1,3,2,4,0)} = Nt_5t_4t_3$ and our relation results in $012 \sim 534$
Thus, our relation becomes $012 \sim 534 \sim 543$.

Applying the transversals to this relation by conjugation, results in the 10 single cosets in the double coset [012]:

$(012 \sim 534 \sim 543)^{Id(N)} = 012 \sim 534 \sim 543$
$(012 \sim 534 \sim 543)^{(5,0)} = 512 \sim 034 \sim 043$
$(012 \sim 534 \sim 543)^{(1,5,4,2)(3,0)} = 351 \sim 302 \sim 320$
$(012 \sim 534 \sim 543)^{(1,2,3,4,5)} = 023 \sim 145 \sim 154$
$(012 \sim 534 \sim 543)^{(1,0,4,3,2)} = 401 \sim 523 \sim 523$
$(012 \sim 534 \sim 543)^{(1,2,3,4,5,0)} = 123 \sim 045 \sim 054$
$(012 \sim 534 \sim 543)^{(1,0,5,4,3,2)} = 501 \sim 423 \sim 432$
$(012 \sim 534 \sim 543)^{(1,3,5,2,4,0)} = 134 \sim 250 \sim 205$
$(012 \sim 534 \sim 543)^{(1,5,4,2,0,3)} = 350 \sim 412 \sim 421$
$(012 \sim 534 \sim 543)^{(1,4)(2,5,3,0)} = 245 \sim 301 \sim 310$

Therefore $[012] =$
$\{Nt_0t_1t_2, Nt_5t_1t_2, Nt_3t_5t_1, Nt_0t_2t_3, Nt_4t_0t_1, Nt_1t_2t_3, Nt_5t_0t_1, Nt_1t_3t_4, Nt_3t_5t_0, Nt_2t_4t_5\}$

Now we must take a representative from each orbit of $N^{(012)}$, so 2, apply each to $Nt_0t_1t_2$ by conjugation, and determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2 \cdot t_2 = Nt_0t_1t_2^2 = Nt_0t_1 \in [01]$
Since the orbit of 2 is of length six,
six t_i's will collapse the Cayley graph from [012] to [01]

Thus, we obtain the following Cayley graph:

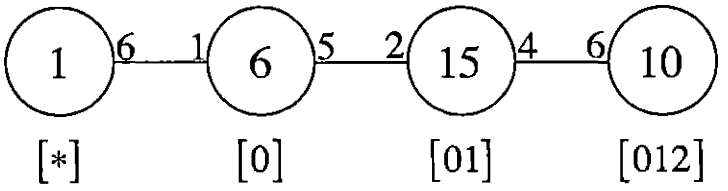

Figure 3.1: The Cayley Graph of $2 \times S_6{:}2^4$ over S_6

So the index of $N = S_6$ in G is $\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(012)}|} = 1+6+15+10 = 32$ Recall that the order of N is $|N| = 6! = 720$. The Cayley diagram gives the maximum order of G, thus $|G| \leq 32 \times |N| = 32 \times 720 = 23040$. In other words, it establishes that $|G| \leq 23040$. Now we must show that $|G| \geq 23040$; thus proving that $|G|$ is actually equal to 23040.

3.2 The Homomorphic Image of G

We begin by labeling our double cosets as shown in the Table 3.1 .

Table 3.1: Labeling of the Double Cosets

Labeling	Double Coset
1	N
2	Nt_0
3	Nt_1
4	Nt_2
5	Nt_3
6	Nt_4
7	Nt_5
8	Nt_0t_1
9	Nt_0t_2
10	Nt_0t_3
11	Nt_1t_2
12	Nt_1t_3
13	Nt_2t_3
14	Nt_2t_4
15	Nt_3t_4
16	Nt_3t_5
17	Nt_4t_0
18	Nt_4t_1
19	Nt_4t_5
20	Nt_5t_0
21	Nt_5t_1
22	Nt_5t_2
23	$Nt_0t_1t_2$
24	$Nt_1t_2t_3$
25	$Nt_5t_0t_1$
26	$Nt_0t_2t_3$
27	$Nt_1t_3t_4$
28	$Nt_2t_4t_5$
29	$Nt_5t_1t_2$
30	$Nt_3t_5t_0$
31	$Nt_3t_5t_1$
32	$Nt_4t_0t_1$

Define $\hat{\alpha} : G \to S_{32}$, where $\hat{\alpha}$ gives the action on the 32 double cosets, this action is by conjugation and multiplication. So, we must compute the action of x, y, and t_0 on the 32 double cosets. These actions are shown in Table 3.2 on the next page.

Table 3.2: Actions on the Double Cosets

Double Coset	tt_0	$xx \sim (012345)$	$yy \sim (01)$
1. N	2. Nt_0	1. N	1. N
2. Nt_0	1. N	3. Nt_1	3. Nt_1
3. Nt_1	8. Nt_0t_1	4. Nt_2	2. Nt_0
4. Nt_2	9. Nt_0t_2	5. Nt_3	4. Nt_2
5. Nt_3	10. Nt_0t_3	6. Nt_4	5. Nt_3
6. Nt_4	17. Nt_4t_0	7. Nt_5	6. Nt_4
7. Nt_5	20. Nt_5t_0	2. Nt_0	7. Nt_5
8. Nt_0t_1	3. Nt_1	11. Nt_1t_2	8. Nt_0t_1
9. Nt_0t_2	4. Nt_2	12. Nt_1t_3	11. Nt_1t_2
10. Nt_0t_3	5. Nt_3	18. Nt_4t_1	12. Nt_1t_3
11. Nt_1t_2	23. $Nt_0t_1t_2$	13. Nt_2t_3	9. Nt_0t_2
12. Nt_1t_3	28. $Nt_2t_4t_5$	14. Nt_2t_4	10. Nt_0t_3
13. Nt_2t_3	26. $Nt_0t_2t_3$	15. Nt_3t_4	13. Nt_2t_3
14. Nt_2t_4	31. $Nt_3t_5t_1$	16. Nt_3t_5	14. Nt_2t_4
15. Nt_3t_4	29. $Nt_5t_1t_2$	19. Nt_4t_5	15. Nt_3t_4
16. Nt_3t_5	30. $Nt_3t_5t_0$	17. Nt_4t_0	16. Nt_3t_5
17. Nt_4t_0	6. Nt_4	21. Nt_5t_1	18. Nt_4t_1
18. Nt_4t_1	32. $Nt_4t_0t_1$	22. Nt_5t_2	17. Nt_4t_0
19. Nt_4t_5	24. $Nt_1t_2t_3$	20. Nt_5t_0	19. Nt_4t_5
20. Nt_5t_0	7. Nt_5	8. Nt_0t_1	21. Nt_5t_1
21. Nt_5t_1	25. $Nt_5t_0t_1$	9. Nt_0t_2	20. Nt_5t_0
22. Nt_5t_2	27. $Nt_1t_3t_4$	10. Nt_0t_3	22. Nt_5t_2
23. $Nt_0t_1t_2$	11. Nt_1t_2	24. $Nt_1t_2t_3$	23. $Nt_0t_1t_2$
24. $Nt_1t_2t_3$	19. Nt_4t_5	25. $Nt_5t_0t_1$	26. $Nt_0t_2t_3$
25. $Nt_5t_0t_1$	21. Nt_5t_1	23. $Nt_0t_1t_2$	25. $Nt_5t_0t_1$
26. $Nt_0t_2t_3$	13. Nt_2t_3	27. $Nt_1t_3t_4$	24. $Nt_1t_2t_3$
27. $Nt_1t_3t_4$	22. Nt_5t_2	28. $Nt_2t_4t_5$	29. $Nt_5t_1t_2$
28. $Nt_2t_4t_5$	12. Nt_1t_3	30. $Nt_3t_5t_0$	28. $Nt_2t_4t_5$
29. $Nt_5t_1t_2$	15. Nt_3t_4	26. $Nt_0t_2t_3$	27. $Nt_1t_3t_4$
30. $Nt_3t_5t_0$	16. Nt_3t_5	32. $Nt_4t_0t_1$	31. $Nt_3t_5t_1$
31. $Nt_3t_5t_1$	14. Nt_2t_4	31. $Nt_3t_5t_1$	30. $Nt_3t_5t_0$
32. $Nt_4t_0t_1$	18. Nt_4t_1	29. $Nt_5t_1t_2$	32. $Nt_4t_0t_1$

Thus we obtain the following information:

Since $\hat{\alpha}(t_0) = tt_0$, then $\hat{\alpha}(t_0)$ is

$\hat{\alpha}(t_0) = (1,2)(3,8)(4,9)(5,10)(6,17)(7,20)(11,23)(12,28)(13,26)(14,31)$
$(15,29)(16,30)(18,32)(19,24)(21,25)(22,27)$

Since $\hat{\alpha}(x) = xx$, then

$\hat{\alpha}(x) = (2,3,4,5,6,7)(8,11,13,15,19,20)(9,12,14,16,17,21)$
$(10,18,22)(23,24,25)(26,27,28,30,32,29)$

Since $\hat{\alpha}(y) = yy$, then

$\hat{\alpha}(y) = (2,3)(9,11)(10,12)(17,18)(20,21)(24,26)(27,29)(30,31)$

Define $\hat{\alpha} : G \to S_{32}$ by $\hat{\alpha}(t_0) = tt_0$, we must show that it is a homomorphism. In order to verify that $\hat{\alpha}$ is a homomorphism we require the following:

(1) If $\hat{\alpha}(N) = \langle xx, yy \rangle$, then $\hat{\alpha}(t_0)$ has exactly six conjugates under conjugation by $\hat{\alpha}(N)$:

We begin by computing $\hat{\alpha}(x)$:

$(t_0)^x = t_1$
$(t_0)^{x^2} = t_2$
$(t_0)^{x^3} = t_3$
$(t_0)^{x^4} = t_4$
$(t_0)^{x^5} = t_5$
$(t_0)^{x^6} = t_0$
$\Rightarrow \hat{\alpha}(t_0^x) = \hat{\alpha}(t_0)^{\hat{\alpha}(x)}$

We must now calculate $tt_1, tt_2, tt_3, tt_4, tt_5$:

$tt_1 = (tt_0)^{xx}$
$= (1,3)(2,8)(4,11)(5,12)(6,18)(7,21)(9,23)(10,28)$
$(13,24)(14,30)(15,27)(16,31)(17,32)(19,26)(20,25)(22,29)$

$tt_2 = (tt_1)^{xx}$
$= (1,4)(2,9)(3,11)(5,13)(6,14)(7,22)(8,23)(10,26)$
$(12,24)(15,25)(16,32)(17,31)(18,30)(19,28)(20,27)(21,29)$

$tt_3 = (tt_2)^{xx}$

$= (1,5)\,(2,10)\,(3,12)\,(4,13)\,(6,15)\,(7,16)\,(8,28)\,(9,26)$

$(11,24)\,(14,25)\,(17,29)\,(18,27)\,(19,23)\,(20,30)\,(21,31)\,(22,32)$

$tt_4 = (tt_3)^{xx}$

$= (1,6)\,(2,17)\,(3,18)\,(4,14)\,(5,15)\,(7,19)\,(8,32)\,(9,31)$

$(10,29)\,(11,30)\,(12,27)\,(13,25)\,(16,23)\,(20,24)\,(21,26)\,(22,28)$

$tt_5 = (tt_4)^{xx}$

$= (1,7)\,(2,20)\,(3,21)\,(4,22)\,(5,16)\,(6,19)\,(8,25)\,(9,27)$

$(10,30)\,(11,29)\,(12,31)\,(13,32)\,(14,28)\,(15,23)\,(17,24)\,(18,26)$

$tt_0 = (tt_5)^{xx}$

$= (1,2)\,(3,8)\,(4,9)\,(5,10)\,(6,17)\,(7,20)\,(11,23)\,(12,28)$

$(13,26)\,(14,31)\,(15,29)\,(16,30)\,(18,32)\,(19,24)\,(21,25)\,(22,27)$

Thus, tt_0 has exactly six conjugates under conjugation by $\langle xx, yy\rangle$.

(2) $\langle xx, yy\rangle$ acts as S_6 on $tt_0, tt_1, tt_2, tt_3, tt_4, tt_5$ by conjugation,
(That is $xx = (tt_0, tt_1, tt_2, tt_3, tt_4, tt_5)$ and $yy = (tt_0, tt_1)$):

We verified above that $xx = (tt_0, tt_1, tt_2, tt_3, tt_4, tt_5)$ by the conjugation of tt_i with xx.

Now, we must verify that $yy = (tt_0, tt_1)$:

First notice:

$(t_0)^y = t_1$
$(t_0)^{y^2} = t_0$
$\Rightarrow \hat{\alpha}(t_0^y) = \hat{\alpha}(t_0)^{\hat{\alpha}(y)}$

Now we must calculate tt_1 and $(tt_1)^{yy}$:

$(tt_1) = (tt_0)^{yy}$
$\Rightarrow tt_1 = (1,3)(2,8)(4,11)(5,12)(6,18)(7,21)(9,23)(10,28)(13,24)(14,30)$
$(15,27)(16,31)(17,32)(19,26)(20,25)(22,29)$

$(tt_1)^{yy} = (1,2)(3,8)(4,9)(5,10)(6,17)(7,20)(11,23)(12,28)(13,26)(14,31)$
$(15,29)(16,30)(18,32)(19,24)(21,25)(22,27)$
$\Rightarrow (tt_1)^{yy} = tt_0$

Therefore $yy = (tt_0, tt_1)$.

Thus, we have shown that $\hat{\alpha}(2^{*6} : S_6) = \hat{\alpha}(N, t_0) =< xx, yy, tt_0 >$ and $< xx, yy, tt_0 >$ is

a homomorphic image of $2^{*6} : S_6$. However, we want the homomorphic image of $G = \frac{2^{*6}:S_6}{[t_0t_1]^2;[(012345)t_0]^6}$. Notice that $< xx, yy, tt_0 >$ is a homomorphic image of G if the additional relations $[t_0t_1]^2 = 1$ and $[(012345)\,t_0]^6 = 1$ hold in $< xx, yy, tt_0 >$.

So, we must verify (i) $\hat{\alpha}(t_0t_1) = \hat{\alpha}(t_1t_0)$ and (ii) $\hat{\alpha}(t_5t_4t_3) = \hat{\alpha}(t_0t_1t_2)$

(i) $\hat{\alpha}(t_0t_1) = \hat{\alpha}(t_1t_0)$:

On the left hand side of the equation, $\hat{\alpha}(t_0t_1) = tt_0tt_1$

$= (1,2)\,(3,8)\,(4,9)\,(5,10)\,(6,17)\,(7,20)\,(11,23)\,(12,28)\,(13,26)\,(14,31)\,(15,29)\,(16,30)$
$(18,32)\,(19,24)\,(21,25)\,(22,27)\,(1,3)\,(2,8)\,(4,11)\,(5,12)\,(6,18)\,(7,21)\,(9,23)\,(10,28)$
$(13,24)\,(14,30)\,(15,27)\,(16,31)\,(17,32)\,(19,26)\,(20,25)\,(22,29)$

$= (1,8)(2,3)(4,23)(5,28)(6,32)(7,25)(9,11)(10,12)(13,19)(14,16)(15,22)(17,18)$
$(20,21)(24,26)(27,29)(30,31)$

On the right hand side of the equation, $\hat{\alpha}(t_1t_0) = tt_1tt_0$

$= (1,3)\,(2,8)\,(4,11)\,(5,12)\,(6,18)\,(7,21)\,(9,23)\,(10,28)\,(13,24)\,(14,30)\,(15,27)\,(16,31)$
$(17,32)\,(19,26)\,(20,25)\,(22,29)\,(1,2)\,(3,8)\,(4,9)\,(5,10)\,(6,17)\,(7,20)\,(11,23)\,(12,28)$
$(13,26)\,(14,31)\,(15,29)\,(16,30)\,(18,32)\,(19,24)\,(21,25)\,(22,27)$

$= (1,8)(2,3)(4,23)(5,28)(6,32)(7,25)(9,11)(10,12)(13,19)(14,16)(15,22)(17,18)$
$(20,21)(24,26)(27,29)(30,31).$

Therefore $\hat{\alpha}(t_0t_1) = \hat{\alpha}(t_1t_0)$.

(ii) $\hat{\alpha}(t_5t_4t_3) = \hat{\alpha}(t_0t_1t_2)$:

On the left hand side of the equation $\hat{\alpha}(t_5t_4t_3) = tt_5tt_4tt_3$

$= (1,7)\,(2,20)\,(3,21)\,(4,22)\,(5,16)\,(6,19)\,(8,25)\,(9,27)\,(10,30)\,(11,29)\,(12,31)\,(13,32)$
$(14,28)\,(15,23)\,(17,24)\,(18,26)\,(1,6)\,(2,17)\,(3,18)\,(4,14)\,(5,15)\,(7,19)\,(8,32)\,(9,31)$
$(10,29)\,(11,30)\,(12,27)\,(13,25)\,(16,23)\,(20,24)\,(21,26)\,(22,28)\,(1,5)\,(2,10)\,(3,12)$
$(4,13)\,(6,15)\,(7,16)\,(8,28)\,(9,26)\,(11,24)\,(14,25)\,(17,29)\,(18,27)\,(19,23)\,(20,30)$
$(21,31)\,(22,32)$

$= (1,23)(2,11)(3,9)(4,8)(5,19)(6,16)(7,15)(10,24)(12,26)(13,28)(14,32)(17,30)$
$(18,31)(20,29)(21,27)(22,25)$

On the right hand side of the equation $\hat{\alpha}(t_0t_1t_2) = tt_0tt_1tt_2$

$= (1,2)\,(3,8)\,(4,9)\,(5,10)\,(6,17)\,(7,20)\,(11,23)\,(12,28)\,(13,26)\,(14,31)\,(15,29)\,(16,30)$
$(18,32)\,(19,24)\,(21,25)\,(22,27)\,(1,3)\,(2,8)\,(4,11)\,(5,12)\,(6,18)\,(7,21)\,(9,23)\,(10,28)$
$(13,24)\,(14,30)\,(15,27)\,(16,31)\,(17,32)\,(19,26)\,(20,25)\,(22,29)\,(1,4)\,(2,9)\,(3,11)$

$(5,13)\,(6,14)\,(7,22)\,(8,23)\,(10,26)\,(12,24)\,(15,25)\,(16,32)\,(17,31)\,(18,30)\,(19,28)$
$(20,27)\,(21,29)$
$= (1,23)(2,11)(3,9)(4,8)(5,19)(6,16)(7,15)(10,24)(12,26)(13,28)(14,32)(17,30)$
$(18,31)(20,29)(21,27)(22,25)$
Therefore $\hat{\alpha}(t_5t_4t_3) = \hat{\alpha}(t_0t_1t_2)$.

Thus, $< xx, yy, tt_0 >$ is a homomorphic image of G. Therefore, $\hat{\alpha}$ preserves the operations of G and $\hat{\alpha} : G \to S_{32}$ is a homomorphism. So, $\hat{\alpha}\,(G) = \langle xx, yy, tt_0 \rangle$ and $|\hat{\alpha}\,(G)| = 23040$.

By the First Isomorphism Theorem, we have that
$\frac{G}{ker\hat{\alpha}} \cong \hat{\alpha}(G)$
$\Rightarrow \frac{|G|}{|ker\hat{\alpha}|} \cong |\langle xx, yy, tt_0 \rangle| = 23040$
$\Rightarrow |G| = 23040 \cdot |ker\hat{\alpha}|$
$\Rightarrow |G| \geq 23040$
But we had previously shown that $|G| \leq 23040$. Therefore $|G| = 23040$

3.3 Permutation and Symmetric Representation

Permutation to Symmetric:

Let $p = (1,22,12,13,8,11,14,19,20,15,16,18)(2,29,31,26)$
$(3,4,28,24,25,23,30,6,7,27,5,32)(9,10,17,21) \in S_{32}$

The goal is to find the symmetric representation form of the permutation p.

By definition, the right cosets of any group, partition the group into disjoint subsets. Hence, any two right cosets of a group will either be equal or disjoint.

Since $Np = Nw$ and $p \in Np$
$\Rightarrow p \in Nw$
$\Rightarrow p = nw$, where $n \in N$ and $N \cong S_4$
In order to find the symmetric representation form of p, we must solve for this n.

$\Rightarrow n = pw^{-1}$.

But $Np = 1^p = 22$ and , from the labeling, $22 = Nt_5t_2$

$\Rightarrow Np = Nt_5t_2$

$\Rightarrow p = nt_5t_2$

$\Rightarrow n = pt_2t_5$

Notice that $pt_2t_5 = (2,3,7)(4,6)(8,21,20)(9,18,22,17,11,19)(10,12,16)$
$(13,15)(23,26,27,32,29,24)(28,31,30)$

Therefore, $n = (2,3,7)(4,6)$

Now we must look at the labeling of the double cosets, shown in Table 1-1, and translate the given permutation:

$\Rightarrow n = (0,1,5)(2,4)$

$\Rightarrow n = (1,5,0)(2,4)$

Thus, $(1,5,0)(2,4)t_5t_2$ is the symmetric representation form of the permutation p.

Chapter 4

The Construction of $A_5 \times A_5 : 2$

There is a computer based proof that a symmetric presentation of $A_5 \times A_5 : 2$ is given by $\frac{2^{*4}:S_4}{[(0123)t_0]^{15};[(01)t_0]^5}$, where $X \sim (0123)$ and $Y \sim (23)$. The progenitor is $2^{*4} : S_4$, and the symmetric presentation of the progenitor is as follows:
$2^{*4} : S_4 \cong \langle x, y, t | x^4 = y^2 = (xy)^3 = 1 = t^2 = [t, y] = [t^x, y] \rangle$.
The control subgroup, N, is S_4. In order to prove that
$(A_5 \times A_5) : 2 \cong \langle x, y, t | x^4 = y^2 = (xy)^3 = 1 = t^2 = [t, y] = [t^x, y] \rangle$, we perform a double coset enumeration of $(A_5 \times A_5) : 2$ over S_4. The double coset enumeration consists of finding all double cosets $[w]$ and figuring out how many single cosets each of these contains. Notice that the double coset enumeration is complete when the set of right cosets is closed under right multiplication. That is, when none of the t_i's can expand the cayley graph any further. From the definition of 2^{*4} we obtain the following property: $t_0^2 = t_1^2 = t_2^2 = t_3^2 = 1$.

The control subgroup is the permutation group, S_4, which consists of the permutations of $0, 1, 2$ and 3, that is $S_4 = \langle (0123), (23) \rangle$. The number of elements contained in S_4, the order of S_4, is calculated as follows: $|S_4| = 4! = 24$. These 24 elements are:
$e, (01), (02), (03), (12), (13), (23), (012), (021), (013), (031), (023), (032), (123),$
$(132), (0123), (0132), (0213), (0231), (0312), (0321), (01)(23), (02)(13), (03)(12)$
We expand our relations below.

The first relation is as follows: $[(01)\, t_0]^5 = e$
Let $\pi = (01)$ .Then by permutation multiplication, we obtain the following:

$\pi^1 = (01) = \pi^3 = \pi^5$

$\pi^2 = e = \pi^4$

So if we expand $[(01)\,t_0]^5 = e$ we obtain the following:

$\pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$

Using the property that $\pi^n \pi^{-n} = e$ we can simplify the expression:

$\Rightarrow \pi\pi^4\pi^{-4} t_0 \pi\pi^3\pi^{-3} t_0 \pi\pi^2\pi^{-2} t_0 \pi\pi^1\pi^{-1} t_0 \pi t_0 = e$

$\Rightarrow \pi^5 \left(\pi^{-4} t_0 \pi^4\right) \left(\pi^{-3} t_0 \pi^3\right) \left(\pi^{-2} t \pi^2\right) \left(\pi^{-1} t \pi^1\right) t_0 = e$

Using the conjugation property, $\pi^{-1} t_i \pi = t_i^{\pi}$

$\Rightarrow \pi^5 t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi^1} t_0 = e$

$\Rightarrow (01)\, t_0^e t_0^{(01)} t_0^e t_0^{(01)} t_0 = e$

$\Rightarrow (01)\, t_0 t_1 t_0 t_1 t_0 = e$

$\Rightarrow (01)\, t_0 t_1 t_0 = t_0 t_1$

Thus, we obtain $(01)\, t_0 t_1 t_0 = t_0 t_1$

The second relation is as follows: $[(0123)\, t_0]^{15} = e$

Let $\pi = (0123)$. Then, by permutation multiplication, we obtain the following:

$\pi^1 = (0123) = \pi^5 = \pi^9 = \pi^{13}$

$\pi^2 = (02)\,(13) = \pi^6 = \pi^{10} = \pi^{14}$

$\pi^3 = (0321) = \pi^7 = \pi^{11} = \pi^{15}$

$\pi^4 = e = \pi^8 = \pi^{12}$

So, if we expand $[(0123)\, t_0]^{15} = e$ we obtain the following:

$\pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$

$\Rightarrow \pi^{15} t_0^{\pi^{14}} t_0^{\pi^{13}} t_0^{\pi^{12}} t_0^{\pi^{11}} t_0^{\pi^{10}} t_0^{\pi^9} t_0^{\pi^8} t_0^{\pi^7} t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi^1} t_0 = e$

$\Rightarrow (0321)\, t_0^{(02)(13)} t_0^{(0123)} t_0^e t_0^{(0321)} t_0^{(02)(13)} t_0^{(0123)} t_0^e$
$t_0^{(0321)} t_0^{(02)(13)} t_0^{(0123)} t_0^e t_0^{(0321)} t_0^{(02)(13)} t_0^{(0123)} t_0 = e$

$\Rightarrow (0321)\, t_2 t_1 t_0 t_3 t_2 t_1 t_0 t_3 t_2 t_1 t_0 t_3 t_2 t_1 t_0 = e$

$\Rightarrow (0321)\, t_2 t_1 t_0 t_3 t_2 t_1 t_0 t_3 = t_0 t_1 t_2 t_3 t_0 t_1 t_2$

Thus, we obtain the relation $(0321)\, t_2 t_1 t_0 t_3 t_2 t_1 t_0 t_3 = t_0 t_1 t_2 t_3 t_0 t_1 t_2$.

4.1 The Double Coset Enumeration of G over S_4

First, note that $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$. We will denote this double coset by $[*]$. The number of elements in $[*]$ is $\frac{|N|}{|N|} = \frac{24}{24} = 1$, hence

$[*]$ consist of the single coset, N. Next, look at the double coset Nt_0 which we will denote by [0]. To obtain the elements in [0], we must first find the point stabiliser of 0, N^0. The point stabiliser consists of permutations in $N = S_4$ which fix 0 and permute $1, 2, 3$. This means that $\left|N^0\right| = 3! = 6$. In this case, we have that the set stabiliser, $N^{(0)} = N^0$. Thus, $\left|N^{(0)}\right| = 3! = 6$. To find the orbits of $N^{(0)}$, we conjugate $0, 1, 2$ and 3 by $N^{(0)}$. The orbits of $N^{(0)}$ are $\{0\}$ and $\{1, 2, 3\}$. The number of single cosets in the double coset [0] is obtained by dividing $\frac{|N|}{\left|N^{(0)}\right|} = \frac{24}{6} = 4$. To obtain the elements in the double coset, [0] we must first find the right coset representatives of $N^{(0)}$ in N, also known as transversals of $N^{(0)}$ in N. The transversals of $N^{(0)}$ in N are $Id(N)$, $(1, 2, 3, 0)$, $(1, 3)(2, 0)$, and (1032). Now we can apply the transversals to Nt_0 by conjugation to obtain the single cosets in [0]:

$N(t_0)^{Id(N)} = Nt_0$

$N(t_0)^{(1,2,3,0)} = Nt_1$

$N(t_0)^{(1,3)(2,0)} = Nt_2$

$N(t_0)^{(1,0,3,2)} = Nt_3$

Therefore $[0] = \{Nt_0, Nt_1, Nt_2, Nt_3\}$.

Now, for the double coset $[*]$ consisting of the single coset N, there are four t_i's that we can apply to N to obtain the right cosets in [0], thus, four t_i's extend the Cayley graph from $[*]$ to [0]. Next we take a representative from each orbit of $N^{(0)}$ and apply it to Nt_0 , by conjugation, to determine if it will extend or collapse the Cayley graph.

$Nt_0 \cdot t_0 = Nt_0^2 = N \in [*]$.
Since the orbit of 0 is of length 1,
one t_i will collapse the Cayley graph from [0] to $[*]$.

$Nt_0 \cdot t_1 = Nt_0t_1$
This is a new double coset which we will label as [01].
Since the orbit of 1 is of length 3,
three t_i's will extend the Cayley graph from [0] to [01].

Now, we consider the double coset Nt_0t_1N which we denoted [01]. To obtain

the elements in [01], we must first find the point stabiliser of 0 and 1, N^{01}. The point stabiliser consists of permutations in $N = S_4$ which fix 0 and 1 and permute 2, 3. The set stabiliser, $N^{(01)} \geq N^{01} = \langle (2,3) \rangle$. In particular, $N^{(01)} = \langle (2,3) \rangle$. In order to find the orbits of $N^{(01)}$ on $\{0,1,2,3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(01)}$. The orbits of $N^{(01)}$ are $\{0\}, \{1\}$ and $\{2,3\}$. The number of single cosets in the double coset [01] is obtained by dividing $\frac{|N|}{|N^{(01)}|} = \frac{24}{2} = 12$. To obtain the elements in this double coset, we must first find the transversals, of $N^{(01)}$ in N. The transversals of $N^{(01)}$ in N are: $Id(N), (1,2,3,0), (1,3)(2,0), (1,3,0), (1,0,3,2), (1,2,0,3), (1,0,2), (1,3,2), (3,0),$ $(1,0), (1,2)$ and $(2,3,0)$.
Now, conjugate Nt_0t_1 by the set stabilizer $N^{(01)}$ to obtain equal names for Nt_0t_1. In this case Nt_0t_1 has only one name, itself.

Now we can apply the transversals to Nt_0t_1 by conjugation to obtain the different single cosets in [01]:

$N(t_0t_1)^{Id(N)} = Nt_0t_1$
$N(t_0t_1)^{(1,2,3,0)} = Nt_1t_2$
$N(t_0t_1)^{(1,3)(2,0)} = Nt_2t_3$
$N(t_0t_1)^{(1,3,0)} = Nt_1t_3$
$N(t_0t_1)^{(1,0,3,2)} = Nt_3t_0$
$N(t_0t_1)^{(1,2,0,3)} = Nt_3t_2$
$N(t_0t_1)^{(1,0,2)} = Nt_2t_0$
$N(t_0t_1)^{(1,3,2)} = Nt_0t_3$
$N(t_0t_1)^{(3,0)} = Nt_3t_1$
$N(t_0t_1)^{(1,0)} = Nt_1t_0$
$N(t_0t_1)^{(1,2)} = Nt_0t_2$
$N(t_0t_1)^{(2,3,0)} = Nt_2t_1$
Therefore, $[01] =$
$\{Nt_0t_1, Nt_0t_2, Nt_0t_3, Nt_1t_0, Nt_1t_2, Nt_1t_3, Nt_2t_0, Nt_2t_1, Nt_2t_3, Nt_3t_0, Nt_3t_1, Nt_3t_2\}$

Now, we must take a representative of each orbit of $N^{(01)}$ and apply it to Nt_0t_1 to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1 \cdot t_0 = Nt_0t_1t_0 \underset{Relation1}{=} Nt_0t_1 \in [01]$.
Since the orbit of 0 is of length 1,
one t_i maps [01] to [01] .

$Nt_0t_1 \cdot t_1 = Nt_0 (t_1)^2 = Nt_0 \in [0]$.
Since the orbit of 1 is of length 1,
one t_i will collapse the cayley graph from [01] to [0].

$Nt_0t_1 \cdot t_2 = Nt_0t_1t_2$.
This is a new double coset which we will label as [012].
Since the orbit of 2 is of length 2,
two t_i's will extend the Cayley graph from [01] to [012].

Now, we look at the double coset $Nt_0t_1t_2N$ which we denoted [012]. To obtain the elements in [012], we must first find the point stabiliser of $0, 1$, and 2, N^{012}. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1$ and 2 and permute 3. The set stabiliser, $N^{(012)} \geq N^{012}$, thus, $\left|N^{(012)}\right| = 1! = 1$. In order to find the orbits of $N^{(012)}$ on $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(012)}$. The orbits of $N^{(012)}$ are $\{0\}, \{1\}, \{2\}$ and $\{3\}$. The number of single cosets in the double coset [012] is obtained by dividing $\frac{|N|}{\left|N^{(012)}\right|} = \frac{24}{1} = 24$. To obtain the elements in this double coset, we must first find the transversals of $N^{(012)}$ in N. We found that the transversals of $N^{(012)}$ in N, are all the elements in S_4.

Now we can apply the transversals to $Nt_0t_1t_2$ by conjugation to obtain the different single cosets in [012]:

$N (t_0t_1t_2)^{Id(N)} = Nt_0t_1t_2$
$N (t_0t_1t_2)^{(1,0)} = Nt_1t_0t_2$
$N (t_0t_1t_2)^{(2,0)} = Nt_2t_1t_0$
$N (t_0t_1t_2)^{(3,0)} = Nt_3t_1t_2$
$N (t_0t_1t_2)^{(1,2)} = Nt_0t_2t_1$
$N (t_0t_1t_2)^{(1,3)} = Nt_0t_3t_2$
$N (t_0t_1t_2)^{(2,3)} = Nt_0t_1t_3$
$N (t_0t_1t_2)^{(0,1,3)} = Nt_1t_3t_2$

$N(t_0t_1t_2)^{(0,3,1)} = Nt_3t_0t_2$
$N(t_0t_1t_2)^{(0,1,2)} = Nt_1t_2t_0$
$N(t_0t_1t_2)^{(0,2,1)} = Nt_2t_0t_1$
$N(t_0t_1t_2)^{(0,2,3)} = Nt_2t_1t_3$
$N(t_0t_1t_2)^{(0,3,2)} = Nt_3t_1t_0$
$N(t_0t_1t_2)^{(1,2,3)} = Nt_0t_2t_3$
$N(t_0t_1t_2)^{(1,3,2)} = Nt_0t_3t_1$
$N(t_0t_1t_2)^{(0,1,2,3)} = Nt_1t_2t_3$
$N(t_0t_1t_2)^{(0,1,3,2)} = Nt_1t_3t_0$
$N(t_0t_1t_2)^{(0,2,1,3)} = Nt_2t_3t_1$
$N(t_0t_1t_2)^{(0,2,3,1)} = Nt_2t_0t_3$
$N(t_0t_1t_2)^{(0,3,1,2)} = Nt_3t_2t_0$
$N(t_0t_1t_2)^{(0,3,2,1)} = Nt_3t_0t_1$
$N(t_0t_1t_2)^{(0,1)(2,3)} = Nt_1t_0t_3$
$N(t_0t_1t_2)^{(0,2)(1,3)} = Nt_2t_3t_0$
$N(t_0t_1t_2)^{(0,3)(1,2)} = Nt_3t_2t_1$
Therefore $[012] =$
$\{Nt_0t_1t_2, Nt_0t_2t_1, Nt_0t_1t_3, Nt_0t_3t_1, Nt_0t_2t_3, Nt_0t_3t_2, Nt_1t_0t_2, Nt_1t_2t_0, Nt_1t_0t_3,$
$Nt_1t_3t_0, Nt_1t_2t_3, Nt_1t_3t_2, Nt_2t_0t_1, Nt_2t_1t_0, Nt_2t_0t_3, Nt_2t_3t_0, Nt_2t_1t_3, Nt_2t_3t_1,$
$Nt_3t_0t_1, Nt_3t_1t_0, Nt_3t_0t_2, , Nt_3t_2t_0, Nt_3t_1t_2, Nt_3t_2t_1\}$

Now, we must take a representative of each orbit of $N^{(012)}$ and apply it to $Nt_0t_1t_2$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2 \cdot t_0 = Nt_0t_1t_2t_0$
This is a new double coset which we will label as [0120].
Since the orbit of 0 is of length 1,
one t_i will extend the Cayley graph from [012] to [0120].

$Nt_0t_1t_2 \cdot t_1 = Nt_0t_1t_2t_1$
But we found that $[0121] \sim [012]$.
Since the orbit of 1 is of length 1,

one t_i maps [012] to [012].

$Nt_0t_1t_2 \cdot t_2 = Nt_0t_1 (t_2)^2 = Nt_0t_1 \in [01]$.
Since the orbit of 2 is of length 1,
one t_i will collapse the Cayley graph from [012] to [01].

$Nt_0t_1t_2 \cdot t_3 = Nt_0t_1t_2t_3$.
This is a new double coset which we will label as [0123].
Since the orbit of 3 is of length 1,
one t_i will extend the Cayley graph from [012] to [0123].

Now, consider the double coset $Nt_0t_1t_2t_3N$ which we denoted [0123]. To obtain the elements in [0123], we must first find the point stabiliser of $0, 1, 2$, and 3, denoted N^{0123}. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1$, 2 and 3. The set stabiliser $N^{(0123)} \geq N^{0123} = \langle (1) \rangle$. Thus, $\left|N^{(0123)}\right| = 1$. To find the orbits of $N^{(0123)}$ on $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(0123)}$. The orbits of $N^{(0123)}$ are $\{0\}, \{1\}, \{2\}$ and $\{3\}$. The number of single cosets in the double coset [0123] is obtained by dividing $\frac{|N|}{\left|N^{(0123)}\right|} = \frac{24}{1} = 24$. To obtain the elements in this double coset, we must first find the transversals of $N^{(0123)}$ in N. The transversals of $N^{(0123)}$ in N are all the elements in S_4.

Now we can apply the transversals to $Nt_0t_1t_2t_3$ by conjugation to obtain the different single cosets in [0123]:

$N (t_0t_1t_2t_3)^{Id(N)} = Nt_0t_1t_2t_3$
$N (t_0t_1t_2t_3)^{(1,0)} = Nt_1t_0t_2t_3$
$N (t_0t_1t_2t_3)^{(2,0)} = Nt_2t_1t_0t_3$
$N (t_0t_1t_2t_3)^{(3,0)} = Nt_3t_1t_2t_0$
$N (t_0t_1t_2t_3)^{(1,2)} = Nt_0t_2t_1t_3$
$N (t_0t_1t_2t_3)^{(1,3)} = Nt_0t_3t_2t_1$
$N (t_0t_1t_2t_3)^{(2,3)} = Nt_0t_1t_3t_2$
$N (t_0t_1t_2t_3)^{(0,1,3)} = Nt_1t_3t_2t_0$
$N (t_0t_1t_2t_3)^{(0,3,1)} = Nt_3t_0t_2t_1$
$N (t_0t_1t_2t_3)^{(0,1,2)} = Nt_1t_2t_0t_3$

$N(t_0t_1t_2t_3)^{(0,2,1)} = Nt_2t_0t_1t_3$
$N(t_0t_1t_2t_3)^{(0,2,3)} = Nt_2t_1t_3t_0$
$N(t_0t_1t_2t_3)^{(0,3,2)} = Nt_3t_1t_0t_2$
$N(t_0t_1t_2t_3)^{(1,2,3)} = Nt_0t_2t_3t_1$
$N(t_0t_1t_2t_3)^{(1,3,2)} = Nt_0t_3t_1t_2$
$N(t_0t_1t_2t_3)^{(0,1,2,3)} = Nt_1t_2t_3t_0$
$N(t_0t_1t_2t_3)^{(0,1,3,2)} = Nt_1t_3t_0t_2$
$N(t_0t_1t_2t_3)^{(0,2,1,3)} = Nt_2t_3t_1t_0$
$N(t_0t_1t_2t_3)^{(0,2,3,1)} = Nt_2t_0t_3t_1$
$N(t_0t_1t_2t_3)^{(0,3,1,2)} = Nt_3t_2t_0t_1$
$N(t_0t_1t_2t_3)^{(0,3,2,1)} = Nt_3t_0t_1t_2$
$N(t_0t_1t_2t_3)^{(0,1)(2,3)} = Nt_1t_0t_3t_2$
$N(t_0t_1t_2t_3)^{(0,2)(1,3)} = Nt_2t_3t_0t_1$
$N(t_0t_1t_2t_3)^{(0,3)(1,2)} = Nt_3t_2t_1t_0$

Therefore [0123] =
$\{Nt_0t_1t_2t_3, Nt_1t_0t_2t_3, Nt_2t_1t_0t_3, Nt_3t_1t_2t_0, Nt_0t_2t_1t_3, Nt_0t_3t_2t_1, Nt_0t_1t_3t_2, Nt_1t_3t_2t_0,$
$Nt_3t_0t_2t_1, Nt_1t_2t_0t_3, Nt_2t_0t_1t_3, Nt_2t_1t_3t_0, Nt_3t_1t_0t_2, Nt_0t_2t_3t_1, Nt_0t_3t_1t_2, Nt_1t_2t_3t_0,$
$Nt_1t_3t_0t_2, Nt_2t_3t_1t_0, Nt_2t_0t_3t_1, Nt_3t_2t_0t_1, Nt_3t_0t_1t_2, Nt_1t_0t_3t_2, Nt_2t_3t_0t_1, Nt_3t_2t_1t_0\}$

Now, we must take a representative of each orbit of $N^{(0123)}$ and apply it to $Nt_0t_1t_2t_3$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3 \cdot t_0 = Nt_0t_1t_2t_3t_0$
This is a new double coset which we will label as [01230].
Since the orbit of 0 is of length 1,
one t_i will extend the Cayley graph from [0123] to [01230].

$Nt_0t_1t_2t_3 \cdot t_1 = Nt_0t_1t_2t_3t_1$
This is a new double coset which we will label as [01231].
Since the orbit of 1 is of length 1,
one t_i will extend the Cayley graph from [0123] to [01231].

$Nt_0t_1t_2t_3 \cdot t_2 = Nt_0t_1t_2t_3t_2$
But we found that $[01232] \sim [0123]$.
Since the orbit of 2 is of length 1,
one t_i will map [0123] to [0123]

$Nt_0t_1t_2t_3 \cdot t_3 = Nt_0t_1t_2\,(t_3)^2 = Nt_0t_1t_2 \in [012]$.
Since the orbit of 3 is of length 1,
one t_i collapse the Cayley graph from [0123] to [012].

Now, consider the double coset $Nt_0t_1t_2t_0N$ which we denoted [0120]. To obtain the elements in [0120], we must first find the point stabiliser of $0,1$ and 2, N^{0120}. Note that in this double coset, we have the relation $[0,1,2,0] \sim [2,1,0,2]$. The point stabiliser consists of permutations in $N = S_4$ which fix $0,1,2$ and permute 3. Hence, the set stabiliser of $0,1$ and 2, $N^{(0120)} \geq N^{0120}$ and, due to the relation mentioned above,the set stabiliser $N^{(0120)}$ increases:

$N\,(t_0t_1t_2t_0)^{(02)} = Nt_2t_1t_0t_2 = Nt_0t_1t_2t_0$, $(\because [0,1,2,0] \sim [2,1,0,2])$
$\Longrightarrow (02) \in N^{(0120)}$
So, $N^{(0120)} = \langle (02) \rangle$

Thus, $\left|N^{(0120)}\right| = 2$. To find the orbits of $N^{(0120)}$ in $\{0,1,2,3\}$, we conjugate $0,1,2$ and 3 by $N^{(0120)}$. The orbits of $N^{(0120)}$ are $\{0,2\}, \{1\}$ and $\{3\}$. The number of single cosets in the double coset [0120] is obtained by dividing $\frac{|N|}{\left|N^{(0120)}\right|} = \frac{24}{2} = 12$. To obtain the elements in this double coset, we must first find the transversals, of $N^{(0120)}$ in N. The transversals of $N^{(0120)}$ in $N$are
$Id(N), (1,2)\,(3,0), (2,3), (1,3), (1,3,0,2), (1,2,0),$
$(1,0,3,2), (1,2,3), (1,0,3), (1,3,2,0), (1,0,2), (3,0).$

Now we can apply the transversals to $Nt_0t_1t_2t_0$ by conjugation to obtain the different single cosets in [0120]:

$N\,(t_0t_1t_2t_0)^{Id(N)} = Nt_0t_1t_2t_0$
$N\,(t_0t_1t_2t_0)^{(1,2)(3,0)} = Nt_3t_2t_1t_3$
$N\,(t_0t_1t_2t_0)^{(2,3)} = Nt_0t_1t_3t_0$
$N\,(t_0t_1t_2t_0)^{(1,3)} = Nt_0t_3t_2t_0$

$N(t_0t_1t_2t_0)^{(1,3,0,2)} = Nt_2t_3t_1t_2$
$N(t_0t_1t_2t_0)^{(1,2,0)} = Nt_1t_2t_0t_1$
$N(t_0t_1t_2t_0)^{(1,0,3,2)} = Nt_3t_0t_1t_3$
$N(t_0t_1t_2t_0)^{(1,2,3)} = Nt_0t_2t_3t_0$
$N(t_0t_1t_2t_0)^{(1,0,3)} = Nt_3t_0t_2t_3$
$N(t_0t_1t_2t_0)^{(1,3,2,0)} = Nt_1t_3t_0t_1$
$N(t_0t_1t_2t_0)^{(1,0,2)} = Nt_2t_0t_1t_2$
$N(t_0t_1t_2t_0)^{(3,0)} = Nt_3t_1t_2t_3$
Therefore $[0120] =$
$\{Nt_0t_1t_2t_0, Nt_3t_2t_1t_3, Nt_0t_1t_3t_0, Nt_0t_3t_2t_0, Nt_2t_3t_1t_2, Nt_1t_2t_0t_1, Nt_3t_0t_1t_3, Nt_0t_2t_3t_0,$
$Nt_3t_0t_2t_3, Nt_1t_3t_0t_1, Nt_2t_0t_1t_2, Nt_3t_1t_2t_3\}$

Now, we must take a representative of each orbit of $N^{(0120)}$ and apply it to $Nt_0t_1t_2t_0$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_0 \cdot t_0 = Nt_0t_1t_2(t_0)^2 = Nt_0t_1t_2 \in [012]$
Since the orbit of 0 is of length 2,
two t_i's will collapse the Cayley graph from [0120] to [012] .

$Nt_0t_1t_2t_0 \cdot t_1 = Nt_0t_1t_2t_0t_1$
Since the orbit of 1 is of length 1,
one t_i will extend the Cayley graph from [0120] to [01201].

$Nt_0t_1t_2t_0 \cdot t_3 = Nt_0t_1t_2t_0t_3$
Since the orbit of 3 is of length 1,
one t_i will extend the Cayley graph from [0120] to [01203]

Now, we look at the double coset $Nt_0t_1t_2t_3t_0N$ which we denoted [01230]. To obtain the elements in [01230], we must first find the point stabiliser of $0, 1, 2$, and 3, N^{01230}. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. The set stabiliser, $N^{(01230)} \geq N^{01230}$ thus, $\left|N^{(01230)}\right| = 1$. In order to find the orbits of $N^{(01230)}$ in $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(01230)}$. The orbits of $N^{(01230)}$ are $\{0\}, \{1\}, \{2\}$ and $\{3\}$. The number of single cosets in the double coset [01230] is

obtained by dividing $\frac{|N|}{|N^{(01230)}|} = \frac{24}{1} = 24$. To obtain the elements in this double coset, we must first find the transversals of $N^{(01230)}$ in N. The transversals of $N^{(01230)}$ in N are all the elements in S_4.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_0$ by conjugation to obtain the different single cosets in [01230]:

$N\,(t_0t_1t_2t_3t_0)^{Id(N)} = Nt_0t_1t_2t_3t_0$
$N\,(t_0t_1t_2t_3t_0)^{(1,0)} = Nt_1t_0t_2t_3t_1$
$N\,(t_0t_1t_2t_3t_0)^{(2,0)} = Nt_2t_1t_0t_3t_2$
$N\,(t_0t_1t_2t_3t_0)^{(3,0)} = Nt_3t_1t_2t_0t_3$
$N\,(t_0t_1t_2t_3t_0)^{(1,2)} = Nt_0t_2t_1t_3t_0$
$N\,(t_0t_1t_2t_3t_0)^{(1,3)} = Nt_0t_3t_2t_1t_0$
$N\,(t_0t_1t_2t_3t_0)^{(2,3)} = Nt_0t_1t_3t_2t_0$
$N\,(t_0t_1t_2t_3t_0)^{(0,1,3)} = Nt_1t_3t_2t_0t_1$
$N\,(t_0t_1t_2t_3t_0)^{(0,3,1)} = Nt_3t_0t_2t_1t_3$
$N\,(t_0t_1t_2t_3t_0)^{(0,1,2)} = Nt_1t_2t_0t_3t_1$
$N\,(t_0t_1t_2t_3t_0)^{(0,2,1)} = Nt_2t_0t_1t_3t_2$
$N\,(t_0t_1t_2t_3t_0)^{(0,2,3)} = Nt_2t_1t_3t_0t_2$
$N\,(t_0t_1t_2t_3t_0)^{(0,3,2)} = Nt_3t_1t_0t_2t_3$
$N\,(t_0t_1t_2t_3t_0)^{(1,2,3)} = Nt_0t_2t_3t_1t_0$
$N\,(t_0t_1t_2t_3t_0)^{(1,3,2)} = Nt_0t_3t_1t_2t_0$
$N\,(t_0t_1t_2t_3t_0)^{(0,1,2,3)} = Nt_1t_2t_3t_0t_1$
$N\,(t_0t_1t_2t_3t_0)^{(0,1,3,2)} = Nt_1t_3t_0t_2t_1$
$N\,(t_0t_1t_2t_3t_0)^{(0,2,1,3)} = Nt_2t_3t_1t_0t_2$
$N\,(t_0t_1t_2t_3t_0)^{(0,2,3,1)} = Nt_2t_0t_3t_1t_2$
$N\,(t_0t_1t_2t_3t_0)^{(0,3,1,2)} = Nt_3t_2t_0t_1t_3$
$N\,(t_0t_1t_2t_3t_0)^{(0,3,2,1)} = Nt_3t_0t_1t_2t_3$
$N\,(t_0t_1t_2t_3t_0)^{(0,1)(2,3)} = Nt_1t_0t_3t_2t_1$
$N\,(t_0t_1t_2t_3t_0)^{(0,2)(1,3)} = Nt_2t_3t_0t_1t_2$
$N\,(t_0t_1t_2t_3t_0)^{(0,3)(1,2)} = Nt_3t_2t_1t_0t_3$

Therefore [01230] =

$\{Nt_0t_1t_2t_3t_0, Nt_1t_0t_2t_3t_1, Nt_2t_1t_0t_3t_2, Nt_3t_1t_2t_0t_3, Nt_0t_2t_1t_3t_0, Nt_0t_3t_2t_1t_0,$
$Nt_0t_1t_3t_2t_0, Nt_1t_3t_2t_0t_1, Nt_3t_0t_2t_1t_3, Nt_1t_2t_0t_3t_1, Nt_2t_0t_1t_3t_2, Nt_2t_1t_3t_0t_2,$
$Nt_3t_1t_0t_2t_3, Nt_0t_2t_3t_1t_0, Nt_0t_3t_1t_2t_0, Nt_1t_2t_3t_0t_1, Nt_1t_3t_0t_2t_1, Nt_2t_3t_1t_0t_2,$
$Nt_2t_0t_3t_1t_2, Nt_3t_2t_0t_1t_3, Nt_3t_0t_1t_2t_3, Nt_1t_0t_3t_2t_1, Nt_2t_3t_0t_1t_2, Nt_3t_2t_1t_0t_3\}$

Now, we must take a representative of each orbit of $N^{(01230)}$ and apply it to $Nt_0t_1t_2t_3t_0$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_0 \cdot t_0 = Nt_0t_1t_2t_3 (t_0)^2 = Nt_0t_1t_2t_3 \in [0123]$
Since the orbit of 0 is of length 1,
one t_i will collapse the Cayley graph from [01230] to [0123].

$Nt_0t_1t_2t_3t_0 \cdot t_1 = Nt_0t_1t_2t_3t_0t_1$
This is a new double coset which we will label as [012301].
Since the orbit of 1 is of length 1,
one t_i will extend the Cayley graph from [01230] to [012301].

$Nt_0t_1t_2t_3t_0 \cdot t_2 = Nt_0t_1t_2t_3t_0t_2$
This is a new double coset which we will label as [012302].
Since the orbit of 2 is of length 1,
one t_i will extend the Cayley graph from [01230] to [012302].

$Nt_0t_1t_2t_3t_0 \cdot t_3 = Nt_0t_1t_2t_3t_0t_3$
But we found that [012303] $\sim$ [01203].
This is a new double coset which we will label as [01203].
Since the orbit of 3 is of length 1,
one t_i maps from [01230] to the new double coset [01203].

Now, we look at the double coset $Nt_0t_1t_2t_3t_1N$ which we denoted [01231]. To obtain the elements in [01231], we must first find the point stabiliser of $0, 1, 2$, and 3, denoted by N^{01231}. Note that for this double coset, we obtained the following relation $[0, 1, 2, 3, 1] \sim [0, 3, 2, 1, 3]$. The point stabiliser consists of the permutations in $N = S_4$

which fix $0, 1, 2$ and 3. The set stabiliser $N^{(01231)} \geq N^{01231}$ and, due to the relation mentioned above, $N^{(01231)}$ increases:

$$N\left(t_0t_1t_2t_3t_1\right)^{(13)} = Nt_0t_3t_2t_1t_3 \underset{relation}{=} Nt_0t_1t_2t_3t_1 \Longrightarrow (13) \in N^{(01231)}$$

So, $N^{(01231)} = \langle (13) \rangle$

Thus, $\left|N^{(01231)}\right| = 2$. In order to find the orbits of $N^{(01231)}$ in $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(01231)}$. The orbits of $N^{(01231)}$ are $\{0\}, \{1, 3\}$ and $\{2\}$. The number of single cosets inside the double coset [01231] is obtained by dividing $\frac{|N|}{\left|N^{(01231)}\right|} = \frac{24}{2} = 12$. To obtain the elements in this double coset, we must first find the transversals of $N^{(01231)}$ in N. The transversals of $N^{(01231)}$ in N are $Id(N), (0,1,2,3), (2,3), (0,2), (0,1,3), (0,1,2), (1,2)(0,3), (0,3,2), (0,2,3,1), (0,2,1,3), (1,2), (0,3,1)$.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_1$ by conjugation to obtain the different single cosets in [01231]:

$N\left(t_0t_1t_2t_3t_1\right)^{Id(N)} = Nt_0t_1t_2t_3t_1$
$N\left(t_0t_1t_2t_3t_1\right)^{(0,1,2,3)} = Nt_1t_2t_3t_0t_2$
$N\left(t_0t_1t_2t_3t_1\right)^{(2,3)} = Nt_0t_1t_3t_2t_1$
$N\left(t_0t_1t_2t_3t_1\right)^{(0,2)} = Nt_2t_1t_0t_3t_1$
$N\left(t_0t_1t_2t_3t_1\right)^{(0,1,3)} = Nt_1t_3t_2t_0t_3$
$N\left(t_0t_1t_2t_3t_1\right)^{(0,1,2)} = Nt_1t_2t_0t_3t_2$
$N\left(t_0t_1t_2t_3t_1\right)^{(1,2)(0,3)} = Nt_3t_2t_1t_0t_2$
$N\left(t_0t_1t_2t_3t_1\right)^{(0,3,2)} = Nt_3t_1t_0t_2t_1$
$N\left(t_0t_1t_2t_3t_1\right)^{(0,2,3,1)} = Nt_2t_0t_3t_1t_0$
$N\left(t_0t_1t_2t_3t_1\right)^{(0,2,1,3)} = Nt_2t_3t_1t_0t_3$
$N\left(t_0t_1t_2t_3t_1\right)^{(1,2)} = Nt_0t_2t_1t_3t_2$
$N\left(t_0t_1t_2t_3t_1\right)^{(0,3,1)} = Nt_3t_0t_2t_1t_0$

Therefore [01231] =
$\{Nt_0t_1t_2t_3t_1, Nt_1t_2t_3t_0t_2, Nt_0t_1t_3t_2t_1, Nt_2t_1t_0t_3t_1, Nt_1t_3t_2t_0t_3, Nt_1t_2t_0t_3t_2,$
$Nt_3t_2t_1t_0t_2, Nt_3t_1t_0t_2t_1, Nt_2t_0t_3t_1t_0, Nt_2t_3t_1t_0t_3, Nt_0t_2t_1t_3t_2, Nt_3t_0t_2t_1t_0\}$

Now, we must take a representative of each orbit of $N^{(01231)}$ and apply it to $Nt_0t_1t_2t_3t_1$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_1 \cdot t_0 = Nt_0t_1t_2t_3t_1t_0$
This is a new double coset which we will label as [012310].
Since the orbit of 0 is of length 1,
one t_i will extend the Cayley graph from [01231] to [012310] .

$Nt_0t_1t_2t_3t_1 \cdot t_1 = Nt_0t_1t_2t_3\,(t_1)^2 = Nt_0t_1t_2t_3 \in [0123]$
Since the orbit of 1 is of length 2,
two t_i's will collapse the Cayley graph from [01231] to [0123].

$Nt_0t_1t_2t_3t_1 \cdot t_2 = Nt_0t_1t_2t_3t_1t_2$
This is a new double coset which we will label as [012312].
Since the orbit of 2 is of length 1,
one t_i will extend the Cayley graph from [01231] to [012312]

Now, we look at the double coset $Nt_0t_1t_2t_0t_1N$, denoted above as [01201]. To obtain the elements in the double coset [01201], we must first find the point stabiliser of $0, 1$ and 2, N^{01201}. The point stabiliser consists of permutations in $N = S_4$ that fix $0, 1$, and 2. The set stabiliser, $N^{(01201)} \geq N^{01201}$. In this double coset, we have the relations $[0,1,2,0,1] \sim [2,0,1,2,0] \sim [2,1,0,2,1] \sim [1,2,0,1,2] \sim [1,0,2,1,0] \sim [0,2,1,0,2]$. Due to this relation, the set stabiliser, $N^{(01201)}$, increases:

$N\,(t_0t_1t_2t_0t_1)^{(02)} = Nt_2t_1t_0t_2t_1 \underset{relation}{=} Nt_0t_1t_2t_0t_1 \Longrightarrow (02) \in N^{(01201)}$
So, $N^{(01231)} \geq \langle (02) \rangle$
$N\,(t_0t_1t_2t_0t_1)^{(021)} = Nt_2t_0t_1t_2t_0 \underset{relation}{=} Nt_0t_1t_2t_0t_1 \Longrightarrow (021) \in N^{(01201)}$
Now, $N^{(01201)} \geq \langle (02), (021) \rangle \cong S_3$.

Thus $N^{(01201)} = S_3$ and therefore $\left|N^{(01201)}\right| = 3! = 6$. In order to find the orbits of $N^{(01201)}$ in $\{0,1,2,3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(01201)}$. The orbits of $N^{(01201)}$ are $\{3\}$ and $\{0,1,2\}$. The number of single cosets in the double coset [01201] is obtained by dividing $\frac{|N|}{\left|N^{(01201)}\right|} = \frac{24}{6} = 4$. To obtain the elements in this double coset, we must first find the transversals of $N^{(01201)}$ in N. The transversals of $N^{(01201)}$ in N are $Id(N)$, $(0,1,2,3), (1,0,2,3), (1,3)\,(0,2)$.

Now we can apply the transversals to $Nt_0t_1t_2t_0t_1$ by conjugation to obtain the different single cosets in [01201]:

$N\left(t_0t_1t_2t_0t_1\right)^{Id(N)} = Nt_0t_1t_2t_0t_1$
$N\left(t_0t_1t_2t_0t_1\right)^{(0,1,2,3)} = Nt_1t_2t_3t_1t_2$
$N\left(t_0t_1t_2t_0t_1\right)^{(1,0,2,3)} = Nt_2t_0t_3t_2t_0$
$N\left(t_0t_1t_2t_0t_1\right)^{(1,3)(0,2)} = Nt_2t_3t_0t_2t_3$
Therefore $[01201] = \{Nt_0t_1t_2t_0t_1, Nt_1t_2t_3t_1t_2, Nt_2t_0t_3t_2t_0, Nt_2t_3t_0t_2t_3\}$

Now, we must take a representative of each orbit of $N^{(01201)}$ and apply it to $Nt_0t_1t_2t_0t_1$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_0t_1 \cdot t_1 = Nt_0t_1t_2t_0\left(t_1\right)^2 = Nt_0t_1t_2t_0 \in [0120]$
Since the orbit of 1 is of length 3,
three t_i's will collapse the Cayley graph from [01201] to [0120].

$Nt_0t_1t_2t_0t_1 \cdot t_3 = Nt_0t_1t_2t_0t_1t_3$
This is a new double coset which we will label as [012013].
Since the orbit of 3 is of length 1,
one t_i will extend the Cayley graph from [01201] to [012013].

Now, we look at the double coset $Nt_0t_1t_2t_0t_3N$ which we denoted [01203], above. To obtain the elements in the double coset [01203], we must first find the point stabiliser of 0, 1, 2 and 3, N^{01203}. The point stabiliser consists of permutations in $N = S_4$ which fix 0, 1, 2 and 3. The set stabiliser $N^{(01203)} \geq N^{01203}$. In this double coset, we have the relation $[0,1,2,0,3] \sim [2,1,0,2,3]$. Due to this relation, the set stabiliser,$N^{(01203)}$, increases:

$N\left(t_0t_1t_2t_0t_3\right)^{(02)} = Nt_2t_1t_0t_2t_3 \underset{relation}{=} Nt_0t_1t_2t_0t_3 \Longrightarrow (02) \in N^{(01203)}$
So, $N^{(01203)} = \langle(02)\rangle$

Thus, $\left|N^{(01203)}\right| = 2! = 2$. In oorder to find the orbits of $N^{(01203)}$, we conjugate $0, 1, 2$ and 3 by $N^{(01203)}$. The orbits of $N^{(01203)}$ are $\{1\}, \{3\}$ and $\{0, 2\}$. The number of single cosets in the double coset [01203] is obtained by dividing $\frac{|N|}{\left|N^{(01203)}\right|} = \frac{24}{2} = 12$.
To obtain the elements in this double coset, we must first find the transversals of $N^{(01203)}$

in N. The transversals of $N^{(01203)}$ in N are $Id(N), (1,2)(0,3), (2,3), (1,3), (1,3,0,2), (1,2,0), (1,0,3,2), (1,2,3), (1,0,3), (1,3,2,0), (1,0,2), (0,3)$

Now we can apply the transversals to $Nt_0t_1t_2t_0t_3$ by conjugation to obtain the different single cosets in [01203]:

$N(t_0t_1t_2t_0t_3)^{Id(N)} = Nt_0t_1t_2t_0t_3$
$N(t_0t_1t_2t_0t_3)^{(1,2)(0.3)} = Nt_3t_2t_1t_3t_0$
$N(t_0t_1t_2t_0t_3)^{(2,3)} = Nt_0t_1t_3t_0t_2$
$N(t_0t_1t_2t_0t_3)^{(1,3)} = Nt_0t_3t_2t_0t_1$
$N(t_0t_1t_2t_0t_3)^{(1,3,0,2)} = Nt_2t_3t_1t_2t_0$
$N(t_0t_1t_2t_0t_3)^{(1,2,0)} = Nt_1t_2t_0t_1t_3$
$N(t_0t_1t_2t_0t_3)^{(1,0,3,2)} = Nt_3t_0t_1t_3t_2$
$N(t_0t_1t_2t_0t_3)^{(1,2,3)} = Nt_0t_2t_3t_0t_1$
$N(t_0t_1t_2t_0t_3)^{(1,0,3)} = Nt_3t_0t_2t_3t_1$
$N(t_0t_1t_2t_0t_3)^{(1,3,2,0)} = Nt_1t_3t_0t_1t_2$
$N(t_0t_1t_2t_0t_3)^{(1,0,2)} = Nt_2t_0t_1t_2t_3$
$N(t_0t_1t_2t_0t_3)^{(0,3)} = Nt_3t_1t_2t_3t_0$

Therefore [01203] =
$\{Nt_0t_1t_2t_0t_3, Nt_3t_2t_1t_3t_0, Nt_0t_1t_3t_0t_2, Nt_0t_3t_2t_0t_1, Nt_2t_3t_1t_2t_0, Nt_1t_2t_0t_1t_3,$
$Nt_3t_0t_1t_3t_2, Nt_0t_2t_3t_0t_1, Nt_3t_0t_2t_3t_1, Nt_1t_3t_0t_1t_2, Nt_2t_0t_1t_2t_3, Nt_3t_1t_2t_3t_0\}$

Now, we must take a representative of each orbit of $N^{(01203)}$ and apply it to $Nt_0t_1t_2t_0t_3$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_0t_3 \cdot t_3 = Nt_0t_1t_2t_0(t_3)^2 = Nt_0t_1t_2t_0 \in [0120]$
Since the orbit of 3 is of length 1,
one t_i will collapse the Cayley graph from [01203] to [0120].

$Nt_0t_1t_2t_0t_3 \cdot t_1 = Nt_0t_1t_2t_0t_3t_1$
This is a new double coset which we will label as [012031].
Since the orbit of 3 is of length 1,
one t_i will extend the Cayley graph from [01201] to [012013].

$Nt_0t_1t_2t_0t_3 \cdot t_2 = Nt_0t_1t_2t_0t_3t_2$
But we found that [012032] $\sim$ [01230]
Since the orbit of 2 is of length 2,
two t_i's will map from [01203] to [01230].

Now, we look at the double coset $Nt_0t_1t_2t_3t_0t_1N$ which we denoted [012301]. To obtain the elements in [012301], we must first find the point stabiliser of $0, 1, 2$, and 3, N^{012301}. The point stabiliser consists of permutations in $N = S_4$ which fix 0, $1, 2$ and 3. The set stabiliser, $N^{(012301)} \geq N^{012301}$. Thus, $\left|N^{(012301)}\right| = 1$. In order to find the orbits of $N^{(012301)}$ in $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(012301)}$. The orbits of $N^{(01230)}$ are $\{0\}, \{1\}, \{2\}$ and $\{3\}$. The number of single cosets in the double coset [012301] is obtained by dividing $\frac{|N|}{\left|N^{(012301)}\right|} = \frac{24}{1} = 24$. To obtain the elements in this double coset, we must first find transversals of $N^{(012301)}$ in N. The transversals of $N^{(012301)}$ in $N$are all the elements in S_4.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_0t_1$ by conjugation to obtain the different single cosets in [012301]:

$N\left(t_0t_1t_2t_3t_0t_1\right)^{Id(N)} = Nt_0t_1t_2t_3t_0t_1$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(1,0)} = Nt_1t_0t_2t_3t_1t_0$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(2,0)} = Nt_2t_1t_0t_3t_2t_1$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(3,0)} = Nt_3t_1t_2t_0t_3t_1$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(1,2)} = Nt_0t_2t_1t_3t_0t_2$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(1,3)} = Nt_0t_3t_2t_1t_0t_3$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(2,3)} = Nt_0t_1t_3t_2t_0t_1$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,1,3)} = Nt_1t_3t_2t_0t_1t_3$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,3,1)} = Nt_3t_0t_2t_1t_3t_0$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,1,2)} = Nt_1t_2t_0t_3t_1t_2$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,2,1)} = Nt_2t_0t_1t_3t_2t_0$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,2,3)} = Nt_2t_1t_3t_0t_2t_1$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,3,2)} = Nt_3t_1t_0t_2t_3t_1$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(1,2,3)} = Nt_0t_2t_3t_1t_0t_2$
$N\left(t_0t_1t_2t_3t_0t_1\right)^{(1,3,2)} = Nt_0t_3t_1t_2t_0t_3$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,1,2,3)} = Nt_1t_2t_3t_0t_1t_2$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,1,3,2)} = Nt_1t_3t_0t_2t_1t_3$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,2,1,3)} = Nt_2t_3t_1t_0t_2t_3$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,2,3,1)} = Nt_2t_0t_3t_1t_2t_0$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,3,1,2)} = Nt_3t_2t_0t_1t_3t_2$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,3,2,1)} = Nt_3t_0t_1t_2t_3t_0$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,1)(2,3)} = Nt_1t_0t_3t_2t_1t_0$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,2)(1,3)} = Nt_2t_3t_0t_1t_2t_3$

$N\left(t_0t_1t_2t_3t_0t_1\right)^{(0,3)(1,2)} = Nt_3t_2t_1t_0t_3t_2$

Therefore [012301] =

$\{Nt_0t_1t_2t_3t_0t_1, Nt_1t_0t_2t_3t_1t_0, Nt_2t_1t_0t_3t_2t_1, Nt_3t_1t_2t_0t_3t_1, Nt_0t_2t_1t_3t_0t_2, Nt_0t_3t_2t_1t_0t_3,$
$Nt_0t_1t_3t_2t_0t_1, Nt_1t_3t_2t_0t_1t_3, Nt_3t_0t_2t_1t_3t_0, Nt_1t_2t_0t_3t_1t_2, Nt_2t_0t_1t_3t_2t_0, Nt_2t_1t_3t_0t_2t_1,$
$Nt_3t_1t_0t_2t_3t_1, Nt_0t_2t_3t_1t_0t_2, Nt_0t_3t_1t_2t_0t_3, Nt_1t_2t_3t_0t_1t_2, Nt_1t_3t_0t_2t_1t_3, Nt_2t_3t_1t_0t_2t_3,$
$Nt_2t_0t_3t_1t_2t_0, Nt_3t_2t_0t_1t_3t_2, Nt_3t_0t_1t_2t_3t_0, Nt_1t_0t_3t_2t_1t_0, Nt_2t_3t_0t_1t_2t_3, Nt_3t_2t_1t_0t_3t_2\}$

Now, we must take a representative of each orbit of $N^{(012301)}$ and apply it to $Nt_0t_1t_2t_3t_0t_1$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_0t_1 \cdot t_0 = Nt_0t_1t_2t_3t_0t_1t_0$
But when checking if this double coset was new,
we found [0123010] $\sim$ [012310].
$\Rightarrow Nt_0t_1t_2t_3t_0t_1t_0 = Nt_0t_1t_2t_3t_1t_0 \in$ [012310]
Since the orbit of 0 is of length 1,
one t_i will map [012301] to the new double coset [012310].

$Nt_0t_1t_2t_3t_0t_1 \cdot t_1 = Nt_0t_1t_2t_3t_0\left(t_1\right)^2 = Nt_0t_1t_2t_3t_0 \in$ [01230]
Since the orbit of 1 is of length 1,
one t_i will collapse the Cayley graph from [012301] to [01230].

$Nt_0t_1t_2t_3t_0t_1 \cdot t_2 = Nt_0t_1t_2t_3t_0t_1t_2$
This is a new double coset which we will label as [0123012].
Since the orbit of 2 is of length 1,

one t_i will extend the Cayley graph from [012301] to [0123012].

$Nt_0t_1t_2t_3t_0t_1 \cdot t_3 = Nt_0t_1t_2t_3t_0t_1t_3$
This is a new double coset which we will label as [0123013].
Since the orbit of 3 is of length 1,
one t_i maps from [012301] to the new double coset [0123013].

Now, we look at the double coset $Nt_0t_1t_2t_3t_0t_2N$ which we denoted [012302]. To obtain the elements in [012302], we must first find the point stabiliser of $0, 1, 2,$ and 3, N^{012302}. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. The set stabiliser, $N^{(012302)} \geq N^{012302}$. In this double coset, we have the relation $[0, 1, 2, 3, 0, 2] \sim [2, 1, 0, 3, 2, 4]$. Due to this relation, the set stabiliser, $N^{(012302)}$, increases:

$N\left(t_0t_1t_2t_3t_0t_2\right)^{(02)} = Nt_2t_1t_0t_3t_2t_0 \underset{relation}{=} Nt_0t_1t_2t_3t_0t_2 \Longrightarrow (02) \in N^{(012302)}$
So, $N^{(012302)} \geq \langle (02) \rangle \cong S_2 \Rightarrow N^{(012302)} = S_2$

Thus, $\left|N^{(012302)}\right| = 2! = 2$. In orer to find the orbits of $N^{(012302)}$ in $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(012302)}$. The orbits of $N^{(012302)}$ are $\{1\}, \{3\}$ and $\{0, 2\}$. The number of single cosets in the double coset [012302] is obtained by dividing $\frac{|N|}{\left|N^{(012302)}\right|} = \frac{24}{2} = 12$. To obtain the elements in this double coset, we must first find the transversals of $N^{(012302)}$ in N. The transversals of $N^{(012302)}$ are: $Id(N), (1, 2)(3, 0), (2, 3), (1, 3), (1, 3, 0, 2), (1, 2, 0), (1, 0, 3, 2), (1, 2, 3), (1, 0, 3), (1, 3, 2, 0), (1, 0, 2), (3, 0)$

Now we can apply the transversals to $Nt_0t_1t_2t_3t_0t_2$ by conjugation to obtain the different single cosets in [012302]:

$N\left(t_0t_1t_2t_3t_0t_2\right)^{Id(N)} = Nt_0t_1t_2t_3t_0t_2$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,2)(3,0)} = Nt_3t_2t_1t_0t_3t_1$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(2,3)} = Nt_0t_1t_3t_2t_0t_3$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,3)} = Nt_0t_3t_2t_1t_0t_2$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,3,0,2)} = Nt_2t_3t_1t_0t_2t_1$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,2,0)} = Nt_1t_2t_0t_3t_1t_0$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,0,3,2)} = Nt_3t_0t_1t_2t_3t_1$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,2,3)} = Nt_0t_2t_3t_1t_0t_3$

$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,0,3)} = Nt_3t_0t_2t_1t_3t_2$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,3,2,0)} = Nt_1t_3t_0t_2t_1t_0$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(1,0,2)} = Nt_2t_0t_1t_3t_2t_1$
$N\left(t_0t_1t_2t_3t_0t_2\right)^{(3,0)} = Nt_3t_1t_2t_0t_3t_2$

Therefore [012302] =
$\{Nt_0t_1t_2t_3t_0t_2, Nt_3t_2t_1t_0t_3t_1, Nt_0t_1t_3t_2t_0t_3, Nt_0t_3t_2t_1t_0t_2, Nt_2t_3t_1t_0t_2t_1, Nt_1t_2t_0t_3t_1t_0,$
$Nt_3t_0t_1t_2t_3t_1, Nt_0t_2t_3t_1t_0t_3, Nt_3t_0t_2t_1t_3t_2, Nt_1t_3t_0t_2t_1t_0, Nt_2t_0t_1t_3t_2t_1, Nt_3t_1t_2t_0t_3t_2\}$

Now, we must take a representative of each orbit of $N^{(012302)}$ and apply it to $Nt_0t_1t_2t_3t_0t_2$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_0t_2 \cdot t_2 = Nt_0t_1t_2t_3t_0\left(t_2\right)^2 = Nt_0t_1t_2t_3t_0$
Since the orbit of 2 is of length 2,
two t_i's will collapse the Cayley graph from [012302] to [01230].

$Nt_0t_1t_2t_3t_0t_2 \cdot t_1 = Nt_0t_1t_2t_3t_0t_2t_1$
This is a new double coset which we will label as [0123021].
Since the orbit of 1 is of length 1,
one t_i will extend the Cayley graph from [012302] to [0123021].

$Nt_0t_1t_2t_3t_0t_2 \cdot t_3 = Nt_0t_1t_2t_3t_0t_2t_3$
But when checking if this double coset was new,
we found [0123023] $\sim$ [012302]
Since the orbit of 3 is of length 1,
one t_i maps from [012302] to the [012302].

Now, we look at the double coset $Nt_0t_1t_2t_3t_1t_0N$ which we denoted [012310]. To obtain the elements in [012310], we must first find the point stabiliser of $0, 1, 2,$ and 3, N^{012310}. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. The set stabiliser, $N^{(012310)} \geq N^{012310}$. In this double coset, we have the relation $[0, 3, 2, 1, 3, 0] \sim [0, 1, 2, 3, 1, 0]$. Due to the relation, the set stabiliser, $N^{(012310)}$, increases:

$$N\left(t_0t_1t_2t_3t_1t_0\right)^{(13)} = Nt_0t_3t_2t_1t_3t_0 \underset{relation}{=} Nt_0t_1t_2t_3t_1t_0 \Longrightarrow (13) \in N^{(012310)}$$

So, $N^{(012310)} \geq \langle (13) \rangle \cong S_2 \Rightarrow N^{(012310)} = S_2$

Thus, $\left|N^{(012310)}\right| = 2! = 2$. In order to find the orbits of $N^{(012310)}$ in $\{0,1,2,3\}$, we conjugate $0,1,2$ and 3 by $N^{(012310)}$. The orbits of $N^{(012310)}$ are $\{0\},\{2\}$ and $\{1,3\}$. The number of single cosets in the double coset [012310] is obtained by dividing $\frac{|N|}{\left|N^{(012310)}\right|} = \frac{24}{2} = 12$. To obtain the elements in this double coset, we must first find the transversals, of $N^{(012310)}$ in N. The transversals of $N^{(012310)}$ in N are: $Id(N), (1,2,3,0), (2,3), (2,0), (1,3,0), (1,2,0), (1,2)(3,0), (2,0,3), (1,0,2,3),$ $(1,3,0,2), (1,2), (1,0,3)$.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_1t_0$ by conjugation to obtain the different single cosets in [012310]:

$N\left(t_0t_1t_2t_3t_1t_0\right)^{Id(N)} = Nt_0t_1t_2t_3t_1t_0$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(1,2,3,0)} = Nt_1t_2t_3t_0t_2t_1$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(2,3)} = Nt_0t_1t_3t_2t_1t_0$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(2,0)} = Nt_2t_1t_0t_3t_1t_2$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(1,3,0)} = Nt_1t_3t_2t_0t_3t_1$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(1,2,0)} = Nt_1t_2t_0t_3t_2t_1$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(1,2)(3,0)} = Nt_3t_2t_1t_0t_2t_3$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(2,0,3)} = Nt_3t_1t_0t_2t_1t_3$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(1,0,2,3)} = Nt_2t_0t_3t_1t_0t_2$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(1,3,0,2)} = Nt_2t_3t_1t_0t_3t_2$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(1,2)} = Nt_0t_2t_1t_3t_2t_0$

$N\left(t_0t_1t_2t_3t_1t_0\right)^{(1,0,3)} = Nt_3t_0t_2t_1t_0t_3$

Therefore [012310] =

$\{Nt_0t_1t_2t_3t_1t_0, Nt_1t_2t_3t_0t_2t_1, Nt_0t_1t_3t_2t_1t_0, Nt_2t_1t_0t_3t_1t_2, Nt_1t_3t_2t_0t_3t_1, Nt_1t_2t_0t_3t_2t_1,$
$Nt_3t_2t_1t_0t_2t_3, Nt_3t_1t_0t_2t_1t_3, Nt_2t_0t_3t_1t_0t_2, Nt_2t_3t_1t_0t_3t_2, Nt_0t_2t_1t_3t_2t_0, Nt_3t_0t_2t_1t_0t_3\}$

Now, we must take a representative of each orbit of $N^{(012310)}$ and apply it to $Nt_0t_1t_2t_3t_1t_0$, to determine if it will extend or collapse the Cayley graph:

$$Nt_0t_1t_2t_3t_1t_0 \cdot t_0 = Nt_0t_1t_2t_3t_1\left(t_0\right)^2 = Nt_0t_1t_2t_3t_1$$

Since the orbit of 0 is of length 1,
one t_i will map from [012310] to [01231] .

$Nt_0t_1t_2t_3t_1t_0 \cdot t_2 = Nt_0t_1t_2t_3t_1t_0t_2$
This is a new double coset which we will label as [0123102].
Since the orbit of 2 is of length 1,
one t_i will extend the Cayley graph from [012310] to [0123102].

$Nt_0t_1t_2t_3t_1t_0 \cdot t_1 = Nt_0t_1t_2t_3t_1t_0t_1$
But when checking if this double coset was new,
we found [0123101] $\sim$ [012301]
Since the orbit of 1 is of length 2,
two t_i's map from [012310] to [012301].

Now, we look at the double coset $Nt_0t_1t_2t_3t_1t_2N$ which we denoted [012312]. To obtain the elements in[012312], we must first find the point stabiliser of $0, 1, 2$, and 3, N^{012312}. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. The set stabiliser, $N^{(012312)} \geq N^{012312}$. In this double coset, we have the relations $[0, 1, 2, 3, 1, 2] \sim [0, 3, 1, 2, 3, 1]$ and $[0, 1, 2, 3, 1, 2] \sim [0, 1, 3, 2, 1, 3]$. Due to this relation, the set stabiliser, $N^{(012312)}$, increases:

$$N\left(t_0t_1t_2t_3t_1t_2\right)^{(132)} = Nt_0t_3t_1t_2t_3t_1 \underset{relation}{=} Nt_0t_1t_2t_3t_1t_2 \Longrightarrow (132) \in N^{(012312)}$$
$$N\left(t_0t_1t_2t_3t_1t_2\right)^{(23)} = Nt_0t_1t_3t_2t_1t_3 \underset{relation}{=} Nt_0t_1t_2t_3t_1t_2 \Longrightarrow (23) \in N^{(012312)}$$
So, $N^{(012310)} \geq \langle (132), (23) \rangle \cong S_3 \Rightarrow N^{(012310)} = S_3$

Thus, $\left|N^{(012312)}\right| = 3! = 6$. In order to find the orbits of $N^{(012312)}$ in $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(012312)}$. The orbits of $N^{(012312)}$ are $\{0\}$ and $\{1, 2, 3\}$. The number of single cosets in the double coset [012312] is obtained by dividing $\frac{|N|}{\left|N^{(012312)}\right|} = \frac{24}{6} = 4$. To obtain the elements in this double coset, we must first find the transversals of $N^{(012312)}$ in N. The transversals of $N^{(012312)}$ in N are: $Id(N), (1, 2, 3, 0), (1, 3)(2, 0), (1, 0, 3, 2)$.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_1t_2N$ by conjugation to obtain the different single cosets in [012312]:

$N\left(t_0t_1t_2t_3t_1t_2\right)^{Id(N)} = Nt_0t_1t_2t_3t_1t_2$
$N\left(t_0t_1t_2t_3t_1t_2\right)^{(1,2,3,0)} = Nt_1t_2t_3t_0t_2t_3$
$N\left(t_0t_1t_2t_3t_1t_2\right)^{(1,3)(2,0)} = Nt_2t_3t_0t_1t_3t_0$
$N\left(t_0t_1t_2t_3t_1t_2\right)^{(1,0,3,2)} = Nt_3t_0t_1t_2t_0t_3$

Thus, $[012312] = \{Nt_0t_1t_2t_3t_1t_2, Nt_1t_2t_3t_0t_2t_3, Nt_2t_3t_0t_1t_3t_0, Nt_3t_0t_1t_2t_0t_3\}$

Now, we must take a representative of each orbit of $N^{(012312)}$ and apply it to $Nt_0t_1t_2t_3t_1t_2$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_1t_2 \cdot t_2 = Nt_0t_1t_2t_3t_1\left(t_2\right)^2 = Nt_0t_1t_2t_3t_1$
Since the orbit of 2 is of length 3,
three t_i's will map from [012312] to [01231].

$Nt_0t_1t_2t_3t_1t_2 \cdot t_0 = Nt_0t_1t_2t_3t_1t_2t_0$
This is a new double coset which we will label as [0123120].
Since the orbit of 0 is of length 1,
one t_i will extend the Cayley graph from [012312] to [0123120].

Now, we look at the double coset $Nt_0t_1t_2t_0t_1t_3N$ which we denoted [012013]. To obtain the elements in [012013], we must first find the point stabiliser of $0, 1, 2$, and 3, N^{012013}. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. The set stabiliser, $N^{(012013)} \geq N^{012013}$. In this double coset, we obtained the following relations $[0, 1, 2, 0, 1, 3] \sim [0, 2, 1, 0, 2, 3]$ and $[0, 1, 2, 0, 1, 3] \sim [1, 2, 0, 1, 2, 3]$. Due to this relation, the set stabiliser, $N^{(012013)}$, increases:

$N\left(t_0t_1t_2t_0t_1t_3\right)^{(012)} = Nt_1t_2t_0t_1t_2t_3 \underset{relation}{=} Nt_0t_1t_2t_0t_1t_3 \Longrightarrow (012) \in N^{(012013)}$
$N\left(t_0t_1t_2t_0t_1t_3\right)^{(12)} = Nt_0t_2t_1t_0t_2t_3 \underset{relation}{=} Nt_0t_1t_2t_0t_1t_3 \Longrightarrow (12) \in N^{(012013)}$
So, $N^{(012013)} \geq \langle (012), (12) \rangle \cong S_3 \Rightarrow N^{(012013)} = S_3$

Thus, $\left|N^{(012013)}\right| = 3! = 6$. In order to find the orbits of $N^{(012013)}$ in $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(012013)}$. The orbits of $N^{(012013)}$ are $\{3\}$ and $\{0, 1, 2\}$. The number of single cosets in the double coset [012013] is obtained by dividing $\frac{|N|}{\left|N^{(012013)}\right|} = \frac{24}{6} = 4$. To obtain the elements in this double coset, we must first find the transversals of $N^{(012013)}$ in N. The transversals of $N^{(012013)}$ in N are: $Id(N), (1, 2, 3, 0), (1, 3)(2, 0)$,

$(1,0,3,2)$

Now we can apply the transversals to $Nt_0t_1t_2t_0t_1t_3$ by conjugation to obtain the distinct single cosets in [012013]:

$N\left(t_0t_1t_2t_0t_1t_3\right)^{Id(N)} = Nt_0t_1t_2t_0t_1t_3$
$N\left(t_0t_1t_2t_0t_1t_3\right)^{(1,2,3,0)} = Nt_1t_2t_3t_1t_2t_0$
$N\left(t_0t_1t_2t_0t_1t_3\right)^{(1,3)(2,0)} = Nt_2t_3t_0t_2t_3t_1$
$N\left(t_0t_1t_2t_0t_1t_3\right)^{(1,0,3,2)} = Nt_3t_0t_1t_3t_0t_2$

Therefore $[012013] = \{Nt_0t_1t_2t_0t_1t_3, Nt_1t_2t_3t_1t_2t_0, Nt_2t_3t_0t_2t_3t_1, Nt_3t_0t_1t_3t_0t_2\}$

Now, we must take a representative of each orbit of $N^{(012013)}$ and apply it to $Nt_0t_1t_2t_0t_1t_3$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_0t_1t_3 \cdot t_3 = Nt_0t_1t_2t_0t_1\left(t_3\right)^2 = Nt_0t_1t_2t_0t_1$
Since the orbit of 3 is of length 1,
one t_i will collapse the Cayley graph from [012013] to [01201] .

$Nt_0t_1t_2t_3t_1t_2 \cdot t_0 = Nt_0t_1t_2t_3t_1t_2t_0$
But when checking if this double coset was new,
we found $[0123120] \sim [012031]$
Since the orbit of 0 is of length 3,
three t_i's map from [012013] to [012031].

Now, we look at the double coset $Nt_0t_1t_2t_0t_3t_1N$ which we denoted [012031]. To obtain the elements in [012031], we must first find the point stabiliser of $0, 1, 2$, and 3, N^{012031}. The point stabiliser consists of permutations in $N = S_4$ which fix 0, 1, 2, and 3. Notice that in this double coset, we have the relation $[0,1,2,0,3,1] \sim [2,1,0,2,3,1]$. Due to the relation mentioned above, the set stabiliser, $N^{(012031)}$, increases:

$N\left(t_0t_1t_2t_0t_3t_1\right)^{(02)} = Nt_2t_1t_0t_2t_3t_1 \underset{relation}{=} Nt_0t_1t_2t_0t_3t_1 \Longrightarrow (02) \in N^{(012031)}$
So, $N^{(012031)} \geq \langle(02)\rangle \cong S_2 \Rightarrow N^{(012031)} = S_2$

Thus, $|N^{(012031)}| = 2! = 2$. In order to find the orbits of $N^{(012031)}$ in $\{0,1,2,3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(012031)}$. The orbits of $N^{(012031)}$ are $\{1\}, \{3\}$ and

$\{0,2\}$. The number of single cosets in the double coset [012031] is obtained by dividing $\frac{|N|}{|N^{(012031)}|} = \frac{24}{2} = 12$. To obtain the elements in this double coset, we must first find the transversals, of $N^{(012031)}$ in N. The transversals of $N^{(012031)}$ in N, are
$Id(N), (1,2)(3,0), (2,3), (1,3), (1,3,0,2),$
$(1,2,0), (1,0,3,2), (1,2,3), (1,0,3), (1,3,2,0), (1,0,2), (3,0)$

Now we can apply the transversals to $Nt_0t_1t_2t_0t_3t_1$ by conjugation to obtain the distinct single cosets in [012031]:

$N(t_0t_1t_2t_0t_3t_1)^{Id(N)} = Nt_0t_1t_2t_0t_3t_1$
$N(t_0t_1t_2t_0t_3t_1)^{(1,2)(3,0)} = Nt_3t_2t_1t_3t_0t_2$
$N(t_0t_1t_2t_0t_3t_1)^{(2,3)} = Nt_0t_1t_3t_0t_2t_1$
$N(t_0t_1t_2t_0t_3t_1)^{(1,3)} = Nt_0t_3t_2t_0t_1t_3$
$N(t_0t_1t_2t_0t_3t_1)^{(1,3,0,2)} = Nt_2t_3t_1t_2t_0t_3$
$N(t_0t_1t_2t_0t_3t_1)^{(1,2,0)} = Nt_1t_2t_0t_1t_3t_2$
$N(t_0t_1t_2t_0t_3t_1)^{(1,0,3,2)} = Nt_3t_0t_1t_3t_2t_0$
$N(t_0t_1t_2t_0t_3t_1)^{(1,2,3)} = Nt_0t_2t_3t_0t_1t_2$
$N(t_0t_1t_2t_0t_3t_1)^{(1,0,3)} = Nt_3t_0t_2t_3t_1t_0$
$N(t_0t_1t_2t_0t_3t_1)^{(0,1,3,2)} = Nt_1t_3t_0t_1t_2t_3$
$N(t_0t_1t_2t_0t_3t_1)^{(1,0,2)} = Nt_2t_0t_1t_2t_3t_0$
$N(t_0t_1t_2t_0t_3t_1)^{(3,0)} = Nt_3t_1t_2t_3t_0t_1$

Therefore [012031] =
$\{Nt_0t_1t_2t_0t_3t_1, Nt_3t_2t_1t_3t_0t_2, Nt_0t_1t_3t_0t_2t_1, Nt_0t_3t_2t_0t_1t_3\ Nt_2t_3t_1t_2t_0t_3, Nt_1t_2t_0t_1t_3t_2,$
$Nt_3t_0t_1t_3t_2t_0, Nt_0t_2t_3t_0t_1t_2 Nt_3t_0t_2t_3t_1t_0, Nt_1t_3t_0t_1t_2t_3, Nt_2t_0t_1t_2t_3t_0, Nt_3t_1t_2t_3t_0t_1\}$

Now, we must take a representative of each orbit of $N^{(012031)}$ and apply it to $Nt_0t_1t_2t_0t_3t_1$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_0t_3t_1 \cdot t_1 = Nt_0t_1t_2t_0t_3 (t_1)^2 = Nt_0t_1t_2t_0t_3$
Since the orbit of 1 is of length 1,
one t_i will collapse the Cayley graph from [012031] to [01203].

$Nt_0t_1t_2t_0t_3t_1 \cdot t_3 = Nt_0t_1t_2t_0t_3t_1t_3$
But when checking if this double coset new,
we found [0120313] $\sim$ [012013]
Since the orbit of 3 is of length 1,
one t_i maps from [012031] to [012013].

$Nt_0t_1t_2t_0t_3t_1 \cdot t_0 = Nt_0t_1t_2t_0t_3t_1t_0$
But when checking if this double coset was new,
we found [0120310] $\sim$ [0123013]
Since the orbit of 0 is of length 2,
two t_i's map from [012031] to [0123013].

Now, we look at the double coset $Nt_0t_1t_2t_3t_0t_1t_2N$ which we denoted [0123012]. To obtain the elements in [0123012], we must first find the point stabiliser of 0, 1, 2, and 3, $N^{0123012}$. The point stabiliser consists of permutations in $N = S_4$ which fix 0, 1, 2 and 3. The set stabiliser, $N^{(0123012)} \geq N^{0123012}$.Thus, $\left|N^{(0123012)}\right| = 1$. In order to find the orbits of $N^{(0123012)}$ in $\{0,1,2,3\}$, we conjugate 0, 1, 2 and 3 by $N^{(0123012)}$. The orbits of $N^{(0123012)}$ are $\{0\},\{1\},\{2\}$ and $\{3\}$. The number of single cosets in the double coset [0123012] is obtained by dividing $\frac{|N|}{\left|N^{(0123012)}\right|} = \frac{24}{1} = 24$. To obtain the elements in this double coset, we must first find the transversals, of $N^{(0123012)}$in N. The transversals of $N^{(0123012)}$ in N, are all the elements in S_4.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_0t_1t_2$ by conjugation to obtain the distinct single cosets in [0123012]:

$N\left(t_0t_1t_2t_3t_0t_1t_2\right)^{Id(N)} = Nt_0t_1t_2t_3t_0t_1t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_2\right)^{(1,0)} = Nt_1t_0t_2t_3t_1t_0t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_2\right)^{(2,0)} = Nt_2t_1t_0t_3t_2t_1t_0$
$N\left(t_0t_1t_2t_3t_0t_1t_2\right)^{(3,0)} = Nt_3t_1t_2t_0t_3t_1t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_2\right)^{(1,2)} = Nt_0t_2t_1t_3t_0t_2t_1$
$N\left(t_0t_1t_2t_3t_0t_1t_2\right)^{(1,3)} = Nt_0t_3t_2t_1t_0t_3t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_2\right)^{(2,3)} = Nt_0t_1t_3t_2t_0t_1t_3$
$N\left(t_0t_1t_2t_3t_0t_1t_2\right)^{(0,1,3)} = Nt_1t_3t_2t_0t_1t_3t_2$

$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,3,1)} = Nt_3t_0t_2t_1t_3t_0t_2$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,1,2)} = Nt_1t_2t_0t_3t_1t_2t_0$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,2,1)} = Nt_2t_0t_1t_3t_2t_0t_1$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,2,3)} = Nt_2t_1t_3t_0t_2t_1t_3$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,3,2)} = Nt_3t_1t_0t_2t_3t_1t_0$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(1,2,3)} = Nt_0t_2t_3t_1t_0t_2t_3$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(1,3,2)} = Nt_0t_3t_1t_2t_0t_3t_1$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,1,2,3)} = Nt_1t_2t_3t_0t_1t_2t_3$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,1,3,2)} = Nt_1t_3t_0t_2t_1t_3t_0$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,2,1,3)} = Nt_2t_3t_1t_0t_2t_3t_1$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,2,3,1)} = Nt_2t_0t_3t_1t_2t_0t_3$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,3,1,2)} = Nt_3t_2t_0t_1t_3t_2t_0$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,3,2,1)} = Nt_3t_0t_1t_2t_3t_0t_1$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,1)(2,3)} = Nt_1t_0t_3t_2t_1t_0t_3$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,2)(1,3)} = Nt_2t_3t_0t_1t_2t_3t_0$
$N(t_0t_1t_2t_3t_0t_1t_2)^{(0,3)(1,2)} = Nt_3t_2t_1t_0t_3t_2t_1$

Therefore [0123012] =
$\{Nt_0t_1t_2t_3t_0t_1t_2, Nt_1t_0t_2t_3t_1t_0t_2, Nt_2t_1t_0t_3t_2t_1t_0, Nt_3t_1t_2t_0t_3t_1t_2, Nt_0t_2t_1t_3t_0t_2t_1,$
$Nt_0t_3t_2t_1t_0t_3t_2, Nt_0t_1t_3t_2t_0t_1t_3, Nt_1t_3t_2t_0t_1t_3t_2, Nt_3t_0t_2t_1t_3t_0t_2, Nt_1t_2t_0t_3t_1t_2t_0,$
$Nt_2t_0t_1t_3t_2t_0t_1, Nt_2t_1t_3t_0t_2t_1t_3, Nt_3t_1t_0t_2t_3t_1t_0, Nt_0t_2t_3t_1t_0t_2t_3, Nt_0t_3t_1t_2t_0t_3t_1,$
$Nt_1t_2t_3t_0t_1t_2t_3, Nt_1t_3t_0t_2t_1t_3t_0, Nt_2t_3t_1t_0t_2t_3t_1, Nt_2t_0t_3t_1t_2t_0t_3, Nt_3t_2t_0t_1t_3t_2t_0,$
$Nt_3t_0t_1t_2t_3t_0t_1, Nt_1t_0t_3t_2t_1t_0t_3, Nt_2t_3t_0t_1t_2t_3t_0, Nt_3t_2t_1t_0t_3t_2t_1\}$

Now, we must take a representative of each orbit of $N^{(0123012)}$ and apply it to $Nt_0t_1t_2t_3t_0t_1t_2$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_0t_1t_2 \cdot t_2 = Nt_0t_1t_2t_3t_0t_1\,(t_2)^2 = Nt_0t_1t_2t_3t_0t_1 \in$ [012301]
Since the orbit of 2 is of length 1,
one t_i will collapse the Cayley graph from [0123012] to [012301] .

$Nt_0t_1t_2t_3t_0t_1t_2 \cdot t_0 = Nt_0t_1t_2t_3t_0t_1t_2t_0$

This is a new double coset which we will label as [01230120].
Since the orbit of 0 is of length 1,
one t_i will extend the Cayley graph from [0123012] to [01230120]

$Nt_0t_1t_2t_3t_0t_1t_2 \cdot t_1 = Nt_0t_1t_2t_3t_0t_1t_2t_1$
When checking if this double coset was new,
we found [01230121] $\sim$ [0123021]
Since the orbit of 1 is of length 1,
one t_i maps from [0123012] to [0123021]

$Nt_0t_1t_2t_3t_0t_1t_2 \cdot t_3 = Nt_0t_1t_2t_3t_0t_1t_2t_3$
When checking if this double coset was new,
we found [01230123] $\sim$ [0123012]
Since the orbit of 3 is of length 1,
one t_i maps from [0123012] to [0123012].

Now, we look at the double coset $Nt_0t_1t_2t_3t_0t_1t_3N$ which we denoted [0123013]. To obtain the elements in [0123013]. we must first find the point stabiliser of $0, 1, 2$, and 3, $N^{0123013}$. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. The set stabiliser, $N^{(0123013)} \geq N^{0123013}$. Thus, $\left|N^{(0123013)}\right| = 1$. In order to find the orbits of $N^{(0123013)}$ in $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(0123013)}$. The orbits of $N^{(0123013)}$ are $\{0\}, \{1\}, \{2\}$ and $\{3\}$. The number of single cosets in the double coset [0123013] is obtained by dividing $\frac{|N|}{\left|N^{(0123013)}\right|} = \frac{24}{1} = 24$. To obtain the elements in this double coset, we must first find the transversals, of $N^{(0123013)}$ in N. The transversals of $N^{(0123013)}$ in N, areall the elements in S_4.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_0t_1t_3$ by conjugation to obtain the distinct single cosets in [0123013]:

$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{Id(N)} = Nt_0t_1t_2t_3t_0t_1t_3$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(1,0)} = Nt_1t_0t_2t_3t_1t_0t_3$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(2,0)} = Nt_2t_1t_0t_3t_2t_1t_3$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(3,0)} = Nt_3t_1t_2t_0t_3t_1t_0$

$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(1,2)} = Nt_0t_2t_1t_3t_0t_2t_3$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(1,3)} = Nt_0t_3t_2t_1t_0t_3t_1$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(2,3)} = Nt_0t_1t_3t_2t_0t_1t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,1,3)} = Nt_1t_3t_2t_0t_1t_3t_0$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,3,1)} = Nt_3t_0t_2t_1t_3t_0t_1$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,1,2)} = Nt_1t_2t_0t_3t_1t_2t_3$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,2,1)} = Nt_2t_0t_1t_3t_2t_0t_3$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,2,3)} = Nt_2t_1t_3t_0t_2t_1t_0$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,3,2)} = Nt_3t_1t_0t_2t_3t_1t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(1,2,3)} = Nt_0t_2t_3t_1t_0t_2t_1$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(1,3,2)} = Nt_0t_3t_1t_2t_0t_3t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,1,2,3)} = Nt_1t_2t_3t_0t_1t_2t_0$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,1,3,2)} = Nt_1t_3t_0t_2t_1t_3t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,2,1,3)} = Nt_2t_3t_1t_0t_2t_3t_0$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,2,3,1)} = Nt_2t_0t_3t_1t_2t_0t_1$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,3,1,2)} = Nt_3t_2t_0t_1t_3t_2t_1$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,3,2,1)} = Nt_3t_0t_1t_2t_3t_0t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,1)(2,3)} = Nt_1t_0t_3t_2t_1t_0t_2$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,2)(1,3)} = Nt_2t_3t_0t_1t_2t_3t_1$
$N\left(t_0t_1t_2t_3t_0t_1t_3\right)^{(0,3)(1,2)} = Nt_3t_2t_1t_0t_3t_2t_0$

Therefore [0123013] =
$\{Nt_0t_1t_2t_3t_0t_1t_3, Nt_1t_0t_2t_3t_1t_0t_3, Nt_2t_1t_0t_3t_2t_1t_3, Nt_3t_1t_2t_0t_3t_1t_0, Nt_0t_2t_1t_3t_0t_2t_3,$
$Nt_0t_3t_2t_1t_0t_3t_1, Nt_0t_1t_3t_2t_0t_1t_2, Nt_1t_3t_2t_0t_1t_3t_0, Nt_3t_0t_2t_1t_3t_0t_1, Nt_1t_2t_0t_3t_1t_2t_3,$
$Nt_2t_0t_1t_3t_2t_0t_3, Nt_2t_1t_3t_0t_2t_1t_0, Nt_3t_1t_0t_2t_3t_1t_2, Nt_0t_2t_3t_1t_0t_2t_1, Nt_0t_3t_1t_2t_0t_3t_2,$
$Nt_1t_2t_3t_0t_1t_2t_0, Nt_1t_3t_0t_2t_1t_3t_2, Nt_2t_3t_1t_0t_2t_3t_0, Nt_2t_0t_3t_1t_2t_0t_1, Nt_3t_2t_0t_1t_3t_2t_1,$
$Nt_3t_0t_1t_2t_3t_0t_2, Nt_1t_0t_3t_2t_1t_0t_2, Nt_2t_3t_0t_1t_2t_3t_1, Nt_3t_2t_1t_0t_3t_2t_0\}$

Now, we must take a representative of each orbit of $N^{(0123013)}$ and apply it to $Nt_0t_1t_2t_3t_0t_1t_3$ to determine if it will extend or collapse the Cayley graph:

$$Nt_0t_1t_2t_3t_0t_1t_3 \cdot t_3 = Nt_0t_1t_2t_3t_0t_1 \left(t_3\right)^2 = Nt_0t_1t_2t_3t_0t_1 \in [012301]$$

Since the orbit of 3 is of length 1,
one t_i will collapse the Cayley graph from [0123013] to [012301].

$Nt_0t_1t_2t_3t_0t_1t_3 \cdot t_0 = Nt_0t_1t_2t_3t_0t_1t_3t_0$
When checking if this double coset was new,
we found [01230130] $\sim$ [0123013]
Since the orbit of 0 is of length 1,
one t_i maps from [0123013] to [0123013]

$Nt_0t_1t_2t_3t_0t_1t_3 \cdot t_1 = Nt_0t_1t_2t_3t_0t_1t_3t_1$
When checking if this double coset was new,
we found [01230131] $\sim$ [012031]
Since the orbit of 1 is of length 1,
one t_i maps from [0123013] to [012031]

$Nt_0t_1t_2t_3t_0t_1t_3 \cdot t_2 = Nt_0t_1t_2t_3t_0t_1t_3t_2$
When checking if this double coset was new,
we found [01230132] $\sim$ [01230120]
Since the orbit of 2 is of length 1,
one t_i maps from [0123013] to [01230120]

Now, we look at the double coset $Nt_0t_1t_2t_3t_0t_2t_1N$ which we denoted [0123021]. To obtain the elements in [0123021]. we must first find the point stabiliser of $0, 1, 2$, and 3, $N^{0123021}$. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. Tthe set stabiliser, $N^{(0123021)} \geq N^{0123021}$.In this double coset, we have the relation $[0, 1, 2, 3, 0, 2, 1] \sim [2, 1, 0, 3, 2, 0, 1]$. Due to this relation, the set stabiliser, $N^{(0123021)}$, increases:

$N\left(t_0t_1t_2t_3t_0t_2t_1\right)^{(02)} = Nt_2t_1t_0t_3t_2t_0t_1 \underset{relation}{=} Nt_0t_1t_2t_3t_0t_2t_1$
$\Longrightarrow (02) \in N^{(0123021)}$
So, $N^{(0123021)} \geq \langle (02) \rangle \cong S_2 \Rightarrow N^{(0123021)} = S_2$

Thus, $\left|N^{(0123021)}\right| = 2! = 2$. In order to find the orbits of $N^{(0123021)}$ in $\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(0123021)}$. The orbits of $N^{(0123021)}$ are $\{1\}, \{3\}$ and

$\{0,2\}$. The number of single cosets in the double coset [0123021] is obtained by dividing $\frac{|N|}{|N^{(0123021)}|} = \frac{24}{2} = 12$. To obtain the elements in this double coset, we must first find the transversals of $N^{(0123021)}$ in N. The transversals of $N^{(0123021)}$ in N are all the elements in S_4.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_0t_2t_1$ by conjugation to obtain the different single cosets in [0123021]:

$N\,(t_0t_1t_2t_3t_0t_2t_1)^{Id(N)} = Nt_0t_1t_2t_3t_0t_2t_1$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(1,0)} = Nt_1t_0t_2t_3t_1t_2t_0$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(2,0)} = Nt_2t_1t_0t_3t_2t_0t_1$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(3,0)} = Nt_3t_1t_2t_0t_3t_2t_1$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(1,2)} = Nt_0t_2t_1t_3t_0t_1t_2$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(1,3)} = Nt_0t_3t_2t_1t_0t_2t_3$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(2,3)} = Nt_0t_1t_3t_2t_0t_3t_1$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,1,3)} = Nt_1t_3t_2t_0t_1t_2t_3$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,3,1)} = Nt_3t_0t_2t_1t_3t_2t_0$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,1,2)} = Nt_1t_2t_0t_3t_1t_0t_2$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,2,1)} = Nt_2t_0t_1t_3t_2t_1t_0$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,2,3)} = Nt_2t_1t_3t_0t_2t_3t_1$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,3,2)} = Nt_3t_1t_0t_2t_3t_0t_1$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(1,2,3)} = Nt_0t_2t_3t_1t_0t_3t_2$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(1,3,2)} = Nt_0t_3t_1t_2t_0t_1t_3$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,1,2,3)} = Nt_1t_2t_3t_0t_1t_3t_2$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,1,3,2)} = Nt_1t_3t_0t_2t_1t_0t_3$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,2,1,3)} = Nt_2t_3t_1t_0t_2t_1t_3$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,2,3,1)} = Nt_2t_0t_3t_1t_2t_3t_0$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,3,1,2)} = Nt_3t_2t_0t_1t_3t_0t_2$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,3,2,1)} = Nt_3t_0t_1t_2t_3t_1t_0$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,1)(2,3)} = Nt_1t_0t_3t_2t_1t_3t_0$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,2)(1,3)} = Nt_2t_3t_0t_1t_2t_0t_3$
$N\,(t_0t_1t_2t_3t_0t_2t_1)^{(0,3)(1,2)} = Nt_3t_2t_{21}t_0t_3t_1t_2$
Therefore, [0123021] =

$\{Nt_0t_1t_2t_3t_0t_2t_1, Nt_1t_0t_2t_3t_1t_2t_0, Nt_2t_1t_0t_3t_2t_0t_1, Nt_3t_1t_2t_0t_3t_2t_1, Nt_0t_2t_1t_3t_0t_1t_2,$
$Nt_0t_3t_2t_1t_0t_2t_3, Nt_0t_1t_3t_2t_0t_3t_1, Nt_1t_3t_2t_0t_1t_2t_3, Nt_3t_0t_2t_1t_3t_2t_0, Nt_1t_2t_0t_3t_1t_0t_2,$
$Nt_2t_0t_1t_3t_2t_1t_0, Nt_2t_1t_3t_0t_2t_3t_1, Nt_3t_1t_0t_2t_3t_0t_1, Nt_0t_2t_3t_1t_0t_3t_2, Nt_0t_3t_1t_2t_0t_1t_3,$
$Nt_1t_2t_3t_0t_1t_3t_2, Nt_1t_3t_0t_2t_1t_0t_3, Nt_2t_3t_1t_0t_2t_1t_3, Nt_2t_0t_3t_1t_2t_3t_0, Nt_3t_2t_0t_1t_3t_0t_2,$
$Nt_3t_0t_1t_2t_3t_1t_0, Nt_1t_0t_3t_2t_1t_3t_0, Nt_2t_3t_0t_1t_2t_0t_3, Nt_3t_2t_{21}t_0t_3t_1t_2\}$

Now, we must take a representative of each orbit of $N^{(0123021)}$ and apply it to $Nt_0t_1t_2t_3t_0t_2t_1$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_0t_2t_1 \cdot t_1 = Nt_0t_1t_2t_3t_0t_2 (t_1)^2 = Nt_0t_1t_2t_3t_0t_2 \in [012302]$
Since the orbit of 1 is of length 1,
one t_i will collapse the Cayley graph from [0123021] to [012302].

$Nt_0t_1t_2t_3t_0t_2t_1 \cdot t_0 = Nt_0t_1t_2t_3t_0t_2t_1t_0$
When checking if this double coset was new,
we found [01230210] $\sim$ [0123012]
Since the orbit of 0 is of length 1,
one t_i maps from [0123021] to [0123012]

$Nt_0t_1t_2t_3t_0t_2t_1 \cdot t_2 = Nt_0t_1t_2t_3t_0t_2t_1t_2$
When checking if this double coset was new,
we found [01230212] $\sim$ [0123012]
Since the orbit of 2 is of length 1,
one t_i maps from [0123021] to [0123012]

$Nt_0t_1t_2t_3t_0t_2t_1 \cdot t_3 = Nt_0t_1t_2t_3t_0t_2t_1t_3$
This is a new double coset which we will label as [01230213].
Since the orbit of 0 is of length 1,
one t_i will extend the Cayley graph from [0123021] to [01230213].

Now, we look at the double coset $Nt_0t_1t_2t_3t_1t_0t_2N$ which we denoted [0123102]. To obtain the elements in [0123102]. we must first find the point stabiliser of $0, 1, 2$, and

3, $N^{0123102}$. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. Hence $\left|N^{0123102}\right| = 1$, and the set stabiliser, $N^{(0123102)} \geq N^{0123102}$. In this double coset, we have the relation $[0,3,2,1,3,0,2] \sim [0,1,2,3,1,0,2]$. Due to the relation, the set stabiliser, $N^{(0123102)}$, increases:

$$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(13)} = Nt_0t_3t_2t_1t_3t_0t_2 \underset{relation}{=} Nt_0t_1t_2t_3t_1t_0t_2$$

$$\Longrightarrow (13) \in N^{(0123102)}$$

So, $N^{(0123102)} \geq \langle(13)\rangle \cong S_2 \Rightarrow N^{(0123102)} = S_2$

Thus, $\left|N^{(0123102)}\right| = 2! = 2$. In order to find the orbits of $N^{(0123102)}$ in $0,1,2,3\}$, we conjugate $0,1,2$ and 3 by $N^{(0123102)}$. The orbits of $N^{(0123102)}$ are $\{0\}, \{2\}$ and $\{1,3\}$. The number of single cosets in the double coset [0123102] is obtained by dividing $\frac{|N|}{\left|N^{(0123102)}\right|} = \frac{24}{2} = 12$. To obtain the elements in this double coset, we must first find the transversals of $N^{(0123102)}$ in N. The transversals of $N^{(0123102)}$ in N are:

$Id(N), (1,2,3,0), (2,3), (2,0), (1,3,0), (1,2,0),$

$(1,2)(3,0), (2,0,3), (1,0,2,3), (1,3,0,2), (1,2), (1,0,3).$

Now we can apply the transversals to $Nt_0t_1t_2t_3t_1t_0t_2$ by conjugation to obtain the different single cosets in [0123102]:

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{Id(N)} = Nt_0t_1t_2t_3t_1t_0t_2$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(1,2,3,0)} = Nt_1t_2t_3t_0t_2t_1t_3$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(2,3)} = Nt_0t_1t_3t_2t_1t_0t_3$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(2,0)} = Nt_2t_1t_0t_3t_1t_2t_0$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(1,3,0)} = Nt_1t_3t_2t_0t_3t_1t_2$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(1,2,0)} = Nt_1t_2t_0t_3t_2t_1t_0$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(1,2)(3,0)} = Nt_3t_2t_1t_0t_2t_3t_1$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(2,0,3)} = Nt_3t_1t_0t_2t_1t_3t_0$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(1,0,2,3)} = Nt_2t_0t_3t_1t_0t_2t_3$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(1,3,0,2)} = Nt_2t_3t_1t_0t_3t_2t_1$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(1,2)} = Nt_0t_2t_1t_3t_2t_0t_1$

$N\left(t_0t_1t_2t_3t_1t_0t_2\right)^{(1,0,3)} = Nt_3t_0t_2t_1t_0t_3t_2$

Therefore [0123102] =

$\{Nt_0t_1t_2t_3t_1t_0t_2, Nt_1t_2t_3t_0t_2t_1t_3, Nt_0t_1t_3t_2t_1t_0t_3, Nt_2t_1t_0t_3t_1t_2t_0, Nt_1t_3t_2t_0t_3t_1t_2,$

$\mathrm{N}t_1t_2t_0t_3t_2t_1t_0, Nt_3t_2t_1t_0t_2t_3t_1, Nt_3t_1t_0t_2t_1t_3t_0, Nt_2t_0t_3t_1t_0t_2t_3, Nt_2t_3t_1t_0t_3t_2t_1,$
$\mathrm{N}t_0t_2t_1t_3t_2t_0t_1, Nt_3t_0t_2t_1t_0t_3t_2\}$

Now, we must take a representative of each orbit of $N^{(0123102)}$ and apply it to $Nt_0t_1t_2t_3t_1t_0t_2$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_1t_0t_2 \cdot t_2 = Nt_0t_1t_2t_3t_1t_0 (t_2)^2 = Nt_0t_1t_2t_3t_1t_0 \in [012310]$
Since the orbit of 2 is of length 1,
one t_i will collapse the Cayley graph from [0123102] to [012310] .

$Nt_0t_1t_2t_3t_1t_0t_2 \cdot t_0 = Nt_0t_1t_2t_3t_1t_0t_2t_0$
When checking if this double coset was new,
we found [01231020] $\sim$ [0123120]
Since the orbit of 0 is of length 1,
one t_i maps from [0123102] to the double coset [0123120]

$Nt_0t_1t_2t_3t_1t_0t_2 \cdot t_1 = Nt_0t_1t_2t_3t_1t_0t_2t_1$
When checking if this double coset was new,
we found [01231021] $\sim$ [01230120]
Since the orbit of 1 is of length 2,
two t_i's map from [0123102] to the new double coset [01230120]

Now, we look at the double coset $Nt_0t_1t_2t_3t_1t_2t_0N$ which we denoted [0123120]. To obtain the elements in [0123120], we must first find the point stabiliser of 0, 1, 2, and 3, $N^{0123120}$. The point stabiliser consists of permutations in $N = S_4$ which fix 0, 1, 2 and 3. The set stabiliser, $N^{(0123120)} \geq N^{0123120}$. In this double coset, we have the following relations $[0,2,3,1,2,3,0] \sim [0,1,2,3,1,2,0]$ and $[0,1,3,2,1,3,0] \sim [0,1,2,3,1,2,0]$. Due to this relation, the set stabiliser, $N^{(0123120)}$, increases:

$N(t_0t_1t_2t_3t_1t_2t_0)^{(123)} = Nt_0t_2t_3t_1t_2t_3t_0 \underset{relation}{=} Nt_0t_1t_2t_3t_1t_2t_0$
$\Longrightarrow (123) \in N^{(0123120)}$
$N(t_0t_1t_2t_3t_1t_2t_0)^{(23)} = Nt_0t_1t_3t_2t_1t_3t_0 \underset{relation}{=} Nt_0t_1t_2t_3t_1t_2t_0$
$\Longrightarrow (23) \in N^{(0123120)}$
So, $N^{(0123120)} \geq \langle (123), (23) \rangle \cong S_3 \Rightarrow N^{(0123120)} = S_3$

Thus, $\left|N^{(0123120)}\right| = 3! = 6$. In order to find the orbits of $N^{(0123120)}$ in $\{0,1,2,3\}$, we conjugate $0,1,2$ and 3 by $N^{(0123120)}$. The orbits of $N^{(0123120)}$ are $\{0\}$ and $\{1,2,3\}$. The number of single cosets in the double coset [0123120] is obtained by dividing $\frac{|N|}{\left|N^{(0123120)}\right|} = \frac{24}{6} = 4$. To obtain the elements in this double coset, we must first find the transversals of $N^{(0123120)}$ in N. The transversals of $N^{(0123120)}$ in N, are: $Id(N), (1,2,3,0), (1,3)(2,0), (1,0,3,2)$

Now we can apply the transversals to $Nt_0t_1t_2t_3t_1t_2t_0$ by conjugation to obtain the distinct single cosets in [0123120]:

$N(t_0t_1t_2t_3t_1t_2t_0)^{Id(N)} = Nt_0t_1t_2t_3t_1t_2t_0$
$N(t_0t_1t_2t_3t_1t_2t_0)^{(1,2,3,0)} = Nt_1t_2t_3t_0t_2t_3t_1$
$N(t_0t_1t_2t_3t_1t_2t_0)^{(1,3)(2,0)} = Nt_2t_0t_2t_1t_3t_0t_2$
$N(t_0t_1t_2t_3t_1t_2t_0)^{(1,0,3,2)} = Nt_3t_0t_1t_2t_0t_1t_3$

Therefore
$[0123102] = \{Nt_0t_1t_2t_3t_1t_2t_0, Nt_1t_2t_3t_0t_2t_3t_1, Nt_2t_0t_2t_1t_3t_0t_2, Nt_3t_0t_1t_2t_0t_1t_3\}$

Now, we must take a representative of each orbit of $N^{(0123120)}$ and apply it to $Nt_0t_1t_2t_3t_1t_2t_0$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_1t_2t_0 \cdot t_0 = Nt_0t_1t_2t_3t_1t_2(t_0)^2 = Nt_0t_1t_2t_3t_1t_2 \in [012312]$
Since the orbit of 0 is of length 1,
one t_i will collapse the Cayley graph from [0123120] to [012312].

$Nt_0t_1t_2t_3t_1t_2t_0 \cdot t_1 = Nt_0t_1t_2t_3t_1t_2t_0t_1$
But when checking if this double coset was new,
we found $[01231201] \sim [0123102]$
Since the orbit of 1 is of length 3,
three t_i's map from [0123120] to the double coset [0123102]

Now, we look at the double coset $Nt_0t_1t_2t_3t_0t_1t_2t_0N$ which we denoted [01230120]. To obtain the elements in [01230120]. we must first find the point stabiliser of $0,1,2$, and 3, $N^{01230120}$. The point stabiliser consists of permutations in $N = S_4$ which fix $0,1,2$ and 3. The set stabiliser, $N^{(01230120)} \geq N^{01230120}$. Thus, $\left|N^{(01230120)}\right| = 1$.

In order to find the orbits of $N^{(01230120)}$, we conjugate $0, 1, 2$ and 3 by $N^{(01230120)}$. The orbits of $N^{(01230120)}$ are $\{0\}, \{1\}, \{2\}$ and $\{3\}$. The number of single cosets in the double coset $[01230120]$ is obtained by dividing $\frac{|N|}{|N^{(01230120)}|} = \frac{24}{1} = 24$. To obtain the elements in this double coset, we must first find the transversals of $N^{(01230120)}$ in N. The transversals of $N^{(01230120)}$ in N are all the elements in S_4.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_0t_1t_2t_0$ by conjugation to obtain the distinct single cosets in [01230120]:

$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{Id(N)} = Nt_0t_1t_2t_3t_0t_1t_2t_0$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(1,0)} = Nt_1t_0t_2t_3t_1t_0t_2t_1$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(2,0)} = Nt_2t_1t_0t_3t_2t_1t_0t_2$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(3,0)} = Nt_3t_1t_2t_0t_3t_1t_2t_3$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(1,2)} = Nt_0t_2t_1t_3t_0t_2t_1t_0$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(1,3)} = Nt_0t_3t_2t_1t_0t_3t_2t_0$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(2,3)} = Nt_0t_1t_3t_2t_0t_1t_3t_0$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,1,3)} = Nt_1t_3t_2t_0t_1t_3t_2t_1$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,3,1)} = Nt_3t_0t_2t_1t_3t_0t_2t_3$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,1,2)} = Nt_1t_2t_0t_3t_1t_2t_0t_1$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,2,1)} = Nt_2t_0t_1t_3t_2t_0t_1t_2$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,2,3)} = Nt_2t_1t_3t_0t_2t_1t_3t_2$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,3,2)} = Nt_3t_1t_0t_2t_3t_1t_0t_3$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(1,2,3)} = Nt_0t_2t_3t_1t_0t_2t_3t_0$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(1,3,2)} = Nt_0t_3t_1t_2t_0t_3t_1t_0$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,1,2,3)} = Nt_1t_2t_3t_0t_1t_2t_3t_1$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,1,3,2)} = Nt_1t_3t_0t_2t_1t_3t_0t_1$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,2,1,3)} = Nt_2t_3t_1t_0t_2t_3t_1t_2$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,2,3,1)} = Nt_2t_0t_3t_1t_2t_0t_3t_2$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,3,1,2)} = Nt_3t_2t_0t_1t_3t_2t_0t_3$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,3,2,1)} = Nt_3t_0t_1t_2t_3t_0t_1t_3$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,1)(2,3)} = Nt_1t_0t_3t_2t_1t_0t_3t_1$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,2)(1,3)} = Nt_2t_3t_0t_1t_2t_3t_0t_2$$
$$N\,(t_0t_1t_2t_3t_0t_1t_2t_0)^{(0,3)(1,2)} = Nt_3t_2t_1t_0t_3t_2t_1t_3$$

Therefore [01230120] =

$\{Nt_0t_1t_2t_3t_0t_1t_2t_0, Nt_1t_0t_2t_3t_1t_0t_2t_1, Nt_2t_1t_0t_3t_2t_1t_0t_2, Nt_3t_1t_2t_0t_3t_1t_2t_3, Nt_0t_2t_1t_3t_0t_2t_1t_0,$
$Nt_0t_3t_2t_1t_0t_3t_2t_0, Nt_0t_1t_3t_2t_0t_1t_3t_0, Nt_1t_3t_2t_0t_1t_3t_2t_1, Nt_3t_0t_2t_1t_3t_0t_2t_3, Nt_1t_2t_0t_3t_1t_2t_0t_1,$
$Nt_2t_0t_1t_3t_2t_0t_1t_2, Nt_2t_1t_3t_0t_2t_1t_3t_2, Nt_3t_1t_0t_2t_3t_1t_0t_3, Nt_0t_2t_3t_1t_0t_2t_3t_0, Nt_0t_3t_1t_2t_0t_3t_1t_0,$
$Nt_1t_2t_3t_0t_1t_2t_3t_1, Nt_1t_3t_0t_2t_1t_3t_0t_1, Nt_2t_3t_1t_0t_2t_3t_1t_2, Nt_2t_0t_3t_1t_2t_0t_3t_2, Nt_3t_2t_0t_1t_3t_2t_0t_3,$
$Nt_3t_0t_1t_2t_3t_0t_1t_3, Nt_1t_0t_3t_2t_1t_0t_3t_1, Nt_2t_3t_0t_1t_2t_3t_0t_2, Nt_3t_2t_1t_0t_3t_2t_1t_3\}$

Now, we must take a representative of each orbit of $N^{(01230120)}$ and apply it to $Nt_0t_1t_2t_3t_0t_1t_2t_0$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2t_3t_0t_1t_2t_0 \cdot t_0 = Nt_0t_1t_2t_3t_0t_1t_2\,(t_0)^2 = Nt_0t_1t_2t_3t_0t_1t_2 \in$ [0123012]
Since the orbit of 0 is of length 1,
one t_i will collapse the Cayley graph from [01230120] to [0123012].

$Nt_0t_1t_2t_3t_0t_1t_2t_0 \cdot t_1 = Nt_0t_1t_2t_3t_0t_1t_2t_0t_1$
When checking if this double coset was new,
we found [012301201] $\sim$ [01230120]
Since the orbit of 1 is of length 1,
one t_i maps from [01230120] to [01230120]

$Nt_0t_1t_2t_3t_0t_1t_2t_0 \cdot t_2 = Nt_0t_1t_2t_3t_0t_1t_2t_0t_2$
When checking if this double coset was new,
we found [012301202] $\sim$ [0123102]
Since the orbit of 2 is of length 1,
one t_i maps from [01230120] to [0123102]

$Nt_0t_1t_2t_3t_0t_1t_2t_0 \cdot t_3 = Nt_0t_1t_2t_3t_0t_1t_2t_0t_3$
When checking if this double coset was new,
we found [012301203] $\sim$ [0123013]
Since the orbit of 3 is of length 1,

one t_i maps from [01230120] to [0123013] .

Now, we look at the double coset $Nt_0t_1t_2t_3t_0t_2t_1t_3N$ which we denoted [01230213]. To obtain the elements in [01230213]. we must first find the point stabiliser of $0, 1, 2,$ and 3, $N^{01230213}$. The point stabiliser consists of permutations in $N = S_4$ which fix $0, 1, 2$ and 3. The set stabiliser, $N^{(01230213)} \geq N^{01230213}$. In this double coset, we have the following relations $[1, 2, 3, 0, 1, 3, 2, 0] \sim [0, 1, 2, 3, 0, 2, 1, 3]$ and $[0, 3, 2, 1, 0, 2, 3, 1] \sim [0, 1, 2, 3, 0, 2, 1, 3]$. Due to this relation, the set stabiliser, $N^{(01230213)}$, increases:

$$N\left(t_0t_1t_2t_3t_0t_2t_1t_3\right)^{(0123)} = Nt_1t_2t_3t_0t_1t_3t_2t_0 \underset{relation}{=} Nt_0t_1t_2t_3t_0t_2t_1t_3$$

$$\Longrightarrow (0123) \in N^{(01230213)}.$$

$$N\left(t_0t_1t_2t_3t_0t_2t_1t_3\right)^{(13)} = Nt_0t_3t_2t_1t_0t_2t_3t_1 \underset{relation}{=} Nt_0t_1t_2t_3t_0t_2t_1t_3$$

$$\Longrightarrow (13) \in N^{(01230213)}$$

So, $N^{(01230213)} \geq \langle (0123), (13) \rangle$

Thus, $\left|N^{(01230213)}\right| = 4\,(2) = 8$. In order to find the orbits of $N^{(01230213)}$ in$\{0, 1, 2, 3\}$, we conjugate $0, 1, 2$ and 3 by $N^{(01230213)}$. The orbits of $N^{(01230213)}$ are $\{0, 1, 2, 3\}$. The number of single cosets in the double coset [01230213] is obtained by dividing $\frac{|N|}{\left|N^{(01230213)}\right|} = \frac{24}{8} = 3$. To obtain the elements in this double coset, we must first find the transversals of $N^{(01230213)}$ in N. The transversals of $N^{(01230213)}$ in N are: $Id(N), (1, 3), (0, 1, 2)$.

Now we can apply the transversals to $Nt_0t_1t_2t_3t_0t_2t_1t_3$ by conjugation to obtain the distinct single cosets in [01230213]:

$$N\left(t_0t_1t_2t_3t_0t_2t_1t_3\right)^{Id(N)} = Nt_0t_1t_2t_3t_0t_2t_1t_3$$
$$N\left(t_0t_1t_2t_3t_0t_2t_1t_3\right)^{(1,3)} = Nt_0t_3t_2t_1t_0t_2t_3t_1$$
$$N\left(t_0t_1t_2t_3t_0t_2t_1t_3\right)^{(0,1,2)} = Nt_1t_2t_0t_3t_1t_0t_2t_3$$

Therefore $[01230213] = \{Nt_0t_1t_2t_3t_0t_2t_1t_3, Nt_0t_3t_2t_1t_0t_2t_3t_1, Nt_1t_2t_0t_3t_1t_0t_2t_3\}$

Now, we must take a representative of each orbit of $N^{(01230213)}$ and apply it to $Nt_0t_1t_2t_3t_0t_2t_1t_3$ to determine if it will extend or collapse the Cayley graph:

$$Nt_0t_1t_2t_3t_0t_2t_1t_3 \cdot t_3 = Nt_0t_1t_2t_3t_0t_2t_1\,(t_3)^2 = Nt_0t_1t_2t_3t_0t_2t_1 \in [0123021]$$

Since the orbit of 3 is of length 4,
all four t_i's will collapse the Cayley graph from [01230213] to [0123021] .

Since the set of right cosets obtained is closed under right coset multiplication,we know that the double coset enumeration is complete. Thus , we obtain the following Cayley Graph:

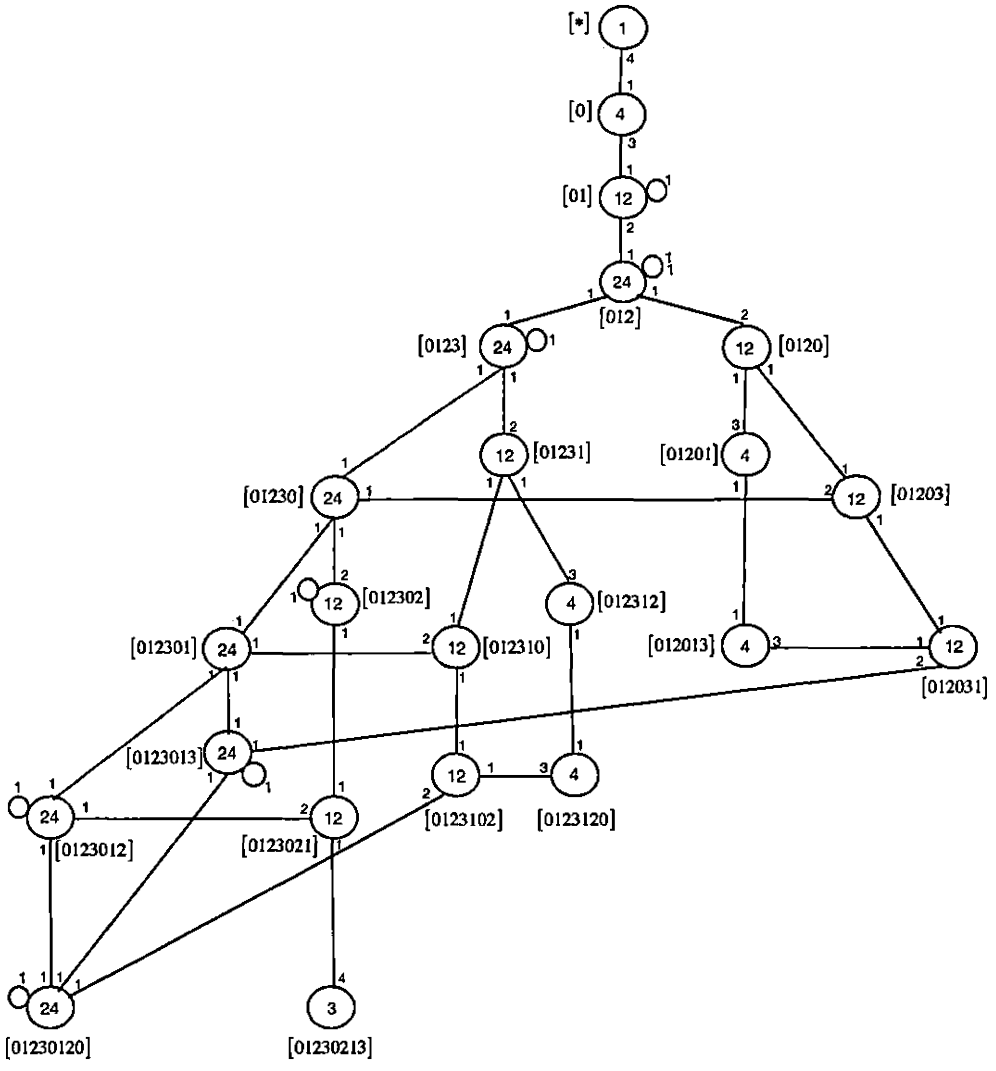

Figure 4.1: The Cayley Graph of $A_5 \times A_5 : 2$ over S_4

Hence, the double coset enumeration shows that the index of $N = S_4$ in G is at most:

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(012)}|} + \frac{|N|}{|N^{(0123)}|} + \frac{|N|}{|N^{(0120)}|} + \frac{|N|}{|N^{(01230)}|} + \frac{|N|}{|N^{(01231)}|} + \frac{|N|}{|N^{(01201)}|} +$$
$$\frac{|N|}{|N^{(01203)}|} + \frac{|N|}{|N^{(012301)}|} + \frac{|N|}{|N^{(012302)}|} + \frac{|N|}{|N^{(012310)}|} + \frac{|N|}{|N^{(012312)}|} + \frac{|N|}{|N^{(012013)}|} + \frac{|N|}{|N^{(012031)}|} +$$
$$\frac{|N|}{|N^{(0123012)}|} + \frac{|N|}{|N^{(0123013)}|} + \frac{|N|}{|N^{(0123021)}|} + \frac{|N|}{|N^{(0123102)}|} + \frac{|N|}{|N^{(0123120)}|} + \frac{|N|}{|N^{(01230120)}|} + \frac{|N|}{|N^{(01230213)}|}$$
$$= 1 + 4 + 12 + 24 + 24 + 12 + 24 + 12 + 4 + 12 + 24 + 12 + 12 + 4 + 4 + 12 + 24 + 24 +$$
$$12 + 12 + 4 + 24 + 3 = 300$$

Recall that the order of N is $|N| = 4! = 24$. Thus $|G| \leq 300 \times |N| = 300 \times 24 = 7200$. Therefore, $|G| \leq 7200$. Now we must show that $|G| \geq 7200$:

4.2 The Homomorphic Image of G

We begin by labeling our double cosets as as shown in Table 4.1:

Table 4.1: Labeling of the Double Cosets

Labeling	Double Coset	Labeling	Double Coset
1	[ ]	35	[2,3,1]
2	[4]	36	[3,2,1,4]
3	[1]	37	[1,4,2]
4	[3]	38	[4,1,3]
5	[2]	39	[4,1,2]
6	[1,4]	40	[1,3,2]
7	[3,4]	41	[1,2,3,4]
8	[2,4]	42	[1,2,4,1]
9	[2,1]	43	[3,4,2]
10	[4,3]	44	[3,1,2]
11	[4,1]	45	[4,2,3,4]
12	[2,3]	46	[1,3,4,1]
13	[3,1]	47	[2,1,3,4]
14	[1,3]	48	[4,3,2,4]
15	[3,2]	49	[2,3,1,4]
16	[2,1,4]	50	[4,3,2,1]
17	[4,2]	51	[2,1,4,3]
18	[1,2]	52	[4,1,3,4]
19	[2,3,4]	53	[4,1,2,4]
20	[3,1,4]	54	[1,3,2,4]
21	[1,3,4]	55	[2,3,4,1]
22	[3,2,4]	56	[4,1,2,3]
23	[3,2,1]	57	[2,3,1,2]
24	[1,4,3]	58	[3,1,2,4]
25	[1,2,4]	59	[3,1,2,3]
26	[3,4,1]	60	[2,4,1,2]
27	[1,2,3]	61	[3,2,4,1]
28	[4,2,1]	62	[1,4,2,3]
29	[2,4,3]	63	[3,4,1,3]
30	[2,4,1]	64	[3,2,1,3]
31	[4,2,3]	65	[3,4,2,1]
32	[4,3,1]	66	[1,2,4,3]
33	[2,1,3]	67	[1,4,3,2]
34	[4,3,2]	68	[4,2,3,1]

Continued on next page

Table 4.1 – *Continued from previous page*

Labeling	Double Coset	Labeling	Double Coset
69	[4,3,2,1,4]	104	[4,3,1,2,4]
70	[3,1,4,2]	105	[4,2,1,3,2]
71	[3,4,2,3]	106	[4,3,2,4,1]
72	[2,4,3,1]	107	[4,1,2,4,3]
73	[4,2,1,3]	108	[4,1,3,2,1]
74	[3,4,1,2]	109	[2,1,4,3,1]
75	[4,1,3,2]	110	[4,1,2,3,1]
76	[4,1,2,3,4]	111	[1,3,4,2,1]
77	[2,3,1,2,4]	112	[3,1,2,4,3]
78	[2,4,1,3]	113	[2,1,4,3,2]
79	[3,1,2,3,4]	114	[1,4,2,3,1]
80	[4,1,2,4,1]	115	[1,4,3,2,1,4]
81	[4,3,1,2]	116	[2,3,1,4,2]
82	[1,3,2,4,3]	117	[3,1,4,2,1]
83	[3,4,1,3,4]	118	[1,3,2,4,1]
84	[3,2,1,3,4]	119	[3,1,4,2,3]
85	[3,4,2,1,4]	120	[1,2,4,3,1]
86	[1,3,4,2]	121	[3,4,2,1,3]
87	[1,2,3,4,2]	122	[2,3,4,1,2]
88	[4,2,3,1,4]	123	[2,4,1,3,2]
89	[1,4,3,2,1]	124	[3,4,1,2,3,4]
90	[3,2,1,4,3]	125	[4,1,3,4,2]
91	[2,3,4,2,3]	126	[3,4,2,3,1,4]
92	[2,4,3,1,4]	127	[1,3,4,1,2]
93	[4,2,1,3,4]	128	[1,2,4,3,2]
94	[3,2,1,4,2]	129	[3,4,1,3,2]
95	[4,1,3,2,4]	130	[2,4,1,2,3,4]
96	[1,2,3,4,1]	131	[1,2,3,1,2,4]
97	[3,4,1,2,3]	132	[3,2,4,1,3]
98	[3,4,2,1,3]	133	[4,2,1,3,2,4]
99	[1,2,4,1,3]	134	[4,1,3,2,1,4]
100	[2,3,1,4,3]	135	[2,3,4,1,3,4]
101	[4,2,3,4,1]	136	[4,1,2,3,1,4]
102	[2,4,1,2,3]	137	[1,3,4,2,1,4]
103	[1,2,3,1,2]	138	[2,1,3,4,2]

Continued on next page

Table 4.1 – *Continued from previous page*

Labeling	Double Coset	Labeling	Double Coset
139	[4,1,2,3,4,2]]	174	[2,1,4,3,2,1,4]
140	[1,4,2,3,1,4]	175	[4,2,3,1,4,2]
141	[2,1,4,3,2,1]	176	[4,1,2,3,1,2]
142	[4,3,2,1,4,3]	177	[2,4,3,1,2,4]
143	[3,4,1,2,4,1]	178	[3,4,2,1,3,2]
144	[2,4,3,1,2]	179	[2,3,1,4,2,1]
145	[4,1,3,2,4,3]	180	[4,1,3,2,4,1]
146	[1,2,4,3,1,4]	181	[2,3,1,4,2,3]
147	[3,4,2,1,3,4]	182	[3,1,2,4,3,1]
148	[4,3,2,1,4,2]	183	[1,3,4,2,1,3]
149	[2,4,1,3,2,4]	184	[1,2,3,4,1,2]
150	[4,1,2,3,4,1]	185	[3,2,4,1,3,2]
151	[2,3,4,1,2,3]	186	[2,3,4,1,2,3,4]
152	[4,1,3,4,2,1]	187	[1,2,4,1,3,2]
153	[2,3,1,2,4,3]	188	[4,1,2,4,3,1]
154	[1,2,3,4,2,3]	189	[2,4,3,1,2,4,1]
155	[3,4,1,3,2,4]	190	[3,2,1,3,4,2]
156	[3,1,2,3,4,1]	191	[4,3,2,4,1,3]
157	[1,3,4,1,2,3]	192	[4,2,3,4,1,2]
158	[2,3,4,2,3,1]	193	[2,4,1,3,2,4,3]
159	[4,1,2,4,1,3]	194	[3,4,1,3,4,2]
160	[4,2,3,1,4,3]	195	[3,1,2,4,3,2]
161	[1,3,2,4,3,1]	196	[2,4,3,1,4,2]
162	[3,1,4,2,1,3]	197	[2,1,4,3,1,2]
163	[1,2,4,3,2,1]	198	[3,1,4,2,1,3,4]
164	[3,4,2,1,4,3]	199	[2,3,1,4,3,2]
165	[1,2,3,4,2,1]	200	[1,2,4,3,2,1,4]
166	[3,2,1,4,2,3]	201	[1,4,2,3,1,2,4]
167	[2,4,1,3,2,1]	202	1,4,3,2,1,3,4[]
168	[3,2,1,4,3,1]	203	[2,1,3,4,2,1,4]
169	[3,4,1,2,3,1]	204	[4,2,1,3,4,2]
170	[2,1,3,4,2,1]	205	[3,2,1,4,3,2,4]
171	[4,3,1,2,4,3]	206	[3,1,4,2,3,1,4]
172	[3,2,1,4,3,2]	207	[3,2,1,4,3,2,1]
173	[3,1,4,2,3,1]	208	[1,4,3,2,1,4,3]

Continued on next page

Table 4.1 – *Continued from previous page*

Labeling	Double Coset	Labeling	Double Coset
209	[4,1,2,3,1,2,4]	244	[1,2,3,4,2,3,1]
210	[1,3,2,4,1,3]	245	[3,4,1,2,4,1,3]
211	[3,4,2,1,3,2,4]	246	[3,1,4,2,3,4,1]
212	[1,2,4,31,2]	247	[2,3,1,4,2,1,3]
213	[2,3,1,4,2,3,4]	248	[1,2,4,3,1,2,4]
214	[3,1,2,4,3,1,4]	249	[1,2,4,3,1,2,3]
215	[1,3,4,2,1,3,4]	250	[4,2,3,1,4,2,1]
216	[1,2,3,4,1,2,4]	251	[2,4,1,3,2,4,1]
217	[3,2,4,1,3,2,4]	252	[4,2,3,1,4,2,3]
218	[3,4,1,2,3,4,1]	253	[2,3,4,1,2,3,1]
219	[1,2,3,4,1,2,3]	254	[4,1,2,3,4,1,3]
220	[3,4,1,2,3,4,2]	255	[4,3,1,2,4,3,1]
221	[3,4,2,1,3,4,1]	256	[2,1,3,4,2,1,3]
222	[3,1,4,2,3,1,2]	257	[4,1,2,3,4,1,2]
223	[1,3,2,4,1,3,4]	258	[1,3,2,4,1,3,2]
224	[1,4,2,3,1,4,3]	259	[4,1,3,2,4,1,2]
225	[1,4,3,2,1,4,2]	260	[4,2,1,3,4,2,3]
226	[1,3,4,2,1,3,2]	261	[2,1,3,4,2,1,3,2]
227	[2,1,4,3,1,2,4]	262	[4,3,1,2,4,3,2]
228	[4,2,1,3,2,4,1]	263	[2,3,1,4,2,3,1,2]
229	[2,4,3,1,4,2,3]	264	[3,2,1,4,2,3,1]
230	[2,3,1,4,3,2,1]	265	[1,2,3,4,2,1,3]
231	[4,1,3,2,1,4,3]	266	[1,3,2,4,3,1,2]
232	[2,1,3,4,2,3,1]	267	[3,4,2,1,4,3,2]
233	[4,3,1,2,4,1,3]	268	[3,2,4,1,3,2,4,3]
234	[2,1,4,3,2,4,1]	269	[1,2,4,3,1,2,4,1]
235	[4,3,2,1,4,2,3]	270	[4,1,2,3,1,4,2]
236	[3,2,4,1,3,2,1]	271	[4,2,3,1,4,3,2]
237	[4,3,2,1,4,3,1]	272	[1,3,4,3,1,4,2,3]
238	[2,1,4,3,2,1,3]	273	[4,2,1,3,4,1,2]
239	[4,2,1,3,4,2,1]	274	[3,2,1,4,3,1,2]
240	[2,4,3,1,2,4,3]	275	[1,2,3,4,1,2,3,1]
241	[4,3,2,1,4,3,2]	276	[3,1,2,4,3,1,2,3]
242	[2,3,1,4,2,3,1]	277	[4,2,1,3,4,2,1,4]
243	[1,4,2,3,1,4,2]	278	[3,4,2,1,3,4,2]

Continued on next page

Table 4.1 – *Continued from previous page*

Labeling	Double Coset	Labeling	Double Coset
279	[4,3,2,1,4,3,2,4]	290	[3,4,2,1,3,4,2,3]
280	[2,3,4,1,3,4,2]	291	[1,3,4,2,1,3,4,1]
281	[4,1,3,2,4,3,1,2]	292	[3,1,4,2,3,1,4,3]
282	[4,1,3,2,4,1,3]	293	[2,1,4,3,2,1,4,2]
283	[3,2,1,4,3,2,1,3]	294	[2,3,4,1,2,3,4,2]
284	[3,1,2,4,3,1,2]	295	[4,1,3,2,4,1,3,4]
285	[4,2,3,1,4,2,3,4]	296	[4,1,2,3,4,2,1,3]
286	[1,3,2,4,1,3,2,1]	297	[2,4,1,3,2,4,1,2]
287	[4,3,1,2,4,3,1,4]	298	[1,4,3,2,1,4,3,1]
288	[4,1,2,3,4,1,2,4]	299	[2,4,3,1,2,4,3,2]
289	[1,4,2,3,1,4,2,1]	300	[3,4,1,2,3,4,1,3]

Define $\hat{\alpha} : G \to S_{300}$ by $\hat{\alpha}(t_0) = tt_0$, where $\hat{\alpha}$ gives the action on the 300 double cosets. Recall that this action is by conjugation and multiplication. We must show that it is a homomorphism. The actions of x, y, and t_0 on the 300 single cosets are as follows:

$$\begin{aligned}\hat{\alpha}(t_0) &= tt_0 \\ &= (1,2)(3,6)(4,7)(5,8)(9,16)(12,19)(13,20)(14,21)(15,22)(18,25)(23,36) \\ &\quad (27,41)(28,42)(31,45)(32,46)(33,47)(34,48)(35,49)(38,52)(39,53) \\ &\quad (40,54)(44,58)(50,69)(56,76)(57,77)(59,79)(60,80)(62,82)(63,83) \\ &\quad (64,84)(65,85)(67,87)(68,88)(71,91)(72,92)(73,93)(74,94)(75,95) \\ &\quad (78,100)(81,104)(89,115)(90,99)(96,101)(97,124)(98,126)(102,130) \\ &\quad (103,131)(105,133)(106,118)(107,112)(108,134)(109,135)(110,136) \\ &\quad (111,137)(113,139)(114,140)(116,127)(117,143)(119,145)(120,146) \\ &\quad (121,147)(122,148)(123,149)(125,138)(128,154)(129,155)(132,160) \\ &\quad (141,174)(142,164)(144,177)(150,161)(151,186)(152,189)(153,159) \\ &\quad (156,158)(157,193)(162,198)(163,200)(165,180)(166,171)(167,201) \\ &\quad (169,202)(170,203)(172,205)(173,206)(175,196)(176,209)(178,211) \\ &\quad (181,213)(182,214)(183,215)(184,216)(185,217)(187,220)(188,221) \\ &\quad (190,194)(191,224)(192,225)(197,227)(199,204)(207,219)(208,233)\end{aligned}$$

$(210,223)(212,248)(218,246)(222,261)(226,263)(228,244)(229,268)$
$(230,269)(231,245)(232,272)(234,251)(235,240)(236,275)(238,276)$
$(239,277)(241,279)(242,258)(243,273)(247,281)(249,283)(252,285)$
$(253,286)(255,287)(256,284)(257,288)(264,291)(265,292)(266,293)$
$(267,294)(270,280)(271,278)(274,296)(282,295)$

$\hat{\alpha}(x) = xx$

$= (2,3,5,4)(6,9,15,10)(7,11,18,12)(8,13,17,14)(16,23,34,24)$
$(19,26,39,27)(20,28,40,29)(21,30,44,31)(22,32,37,33)(25,35,43,38)$
$(36,50,67,51)(41,55,74,56)(42,57,71,52)(45,46,60,59)(47,61,81,62)$
$(76,96,122,97)(48,63,53,64)(49,65,75,66)(54,72,70,73)(58,68,86,78)$
$(69,89,113,90)(77,98,125,99)(79,101,127,102)(80,103,91,83)$
$(82,92,117,105)(84,106,129,107)(85,108,128,100)(87,109,94,110)$
$(88,111,123,112)(93,118,144,119)(95,120,116,121)(104,114,138,132)$
$(115,141,172,142)(124,150,184,151)(126,152,187,153)(130,156,192,157)$
$(131,158,194,159)(133,161,196,162)(134,163,199,164)(135,143,176,154)$
$(136,165,197,166)(137,167,195,160)(139,168,148,169)(140,170,185,171)$
$(145,146,179,178)(147,180,212,181)(149,182,175,183)(155,188,190,191)$
$(173,204,210,177)(174,207,241,208)(186,218,257,219)(189,222,260,223)$
$(193,214,250,226)(198,228,266,229)(200,230,267,231)(201,232,271,233)$
$(202,234,274,235)(203,236,262,224)(205,237,225,238)(206,239,258,240)$
$(209,244,280,245)(211,246,273,247)(213,221,259,249)(215,251,284,252)$
$(216,253,220,254)(217,255,243,256)(227,264,270,265)(242,278,282,248)$
$(261,268,287,289)(263,290,295,269)(272,281)(275,294,300,288)$
$(276,285,291,297)(277,286,299,292)(279,298,293,283)$

$\hat{\alpha}(y) = yy$

$= (4,5)(7,8)(9,13)(10,17)(12,15)(14,18)(16,20)(19,22)(21,25)$
$(23,35)(24,37)(26,30)(27,40)(28,32)(29,43)(31,34)(33,44)(36,49)$
$(38,39)(41,54)(42,46)(45,48)(47,58)(50,68)(51,70)(52,53)(55,61)$
$(56,75)(57,64)(60,63)(62,67)(65,72)(66,86)(69,88)(73,81)(74,78)$
$(76,95)(77,84)(80,83)(82,87)(85,92)(89,114)(90,116)(93,104)$

$(94,100)(96,118)(97,123)(99,127)(101,106)(102,129)(107,125)$
$(108,110)(109,117)(111,120)(112,138)(113,119)(115,140)(121,144)$
$(122,132)(124,149)(130,155)(134,136)(135,143)(137,146)(139,145)$
$(141,173)(142,175)(147,177)(148,160)(150,180)(151,185)(152,188)$
$(153,190)(157,187)(159,194)(161,165)(162,197)(164,196)(166,199)$
$(167,169)(168,179)(170,182)(171,204)(172,181)(174,206)(183,212)$
$(184,210)(186,217)(189,221)(191,192)(193,220)(198,227)(201,202)$
$(203,214)(205,213)(207,242)(208,243)(215,248)(216,223)(218,251)$
$(219,258)(222,238)(224,225)(226,249)(229,267)(230,264)(231,270)$
$(233,273)(234,246)(235,271)(236,253)(237,250)(239,255)(240,278)$
$(241,252)(245,280)(247,274)(254,259)(256,284)(257,282)(260,262)$
$(261,276)(263,283)(265,266)(268,294)(269,291)(275,286)(277,287)$
$(279,285)(281,296)(288,295)(289,298)(290,299)(292,293)(297,300)$

In order to verify the $\hat{\alpha}$ is a homomorphism, we require the following:

(1) If $\hat{\alpha}(N) = \langle xx, yy \rangle$, then $\hat{\alpha}(t_0)$ has exactly four conjugates under conjugation by $\hat{\alpha}(N)$:

We begin by computing $\hat{\alpha}(x)$:
$(t_0)^x = t_1$
$(t_0)^{x^2} = t_2$
$(t_0)^{x^3} = t_3$
$(t_0)^{x^4} = t_0$
$\Rightarrow \hat{\alpha}(t_0^x) = \hat{\alpha}(t_0)^{\hat{\alpha}(x)}$
We must now calculate tt_1, tt_2, tt_3:

$tt_1 = (tt_0)^{xx}$
$= (1,3)(2,11)(4,13)(5,9)(7,26)(8,30)(10,32)(12,35)(15,23)(17,28)$
$(1955)(21,46)(22,61)(24,63)(25,42)(27,64)(29,72)(31,68)(34,50)$
$(37,60)(40,57)(41,96)(43,65)(45,101)(47,92)(48,106)(51,109)$
$(52,83)(53,80)(54,118)(56,110)(58,85)(59,103)(62,114)(66,120)$

$(67,89)(69,77)(70,117)(71,98)(75,108)(76,150)(79,156)(82,161)$
$(84,88)(86,111)(87,165)(90,168)(91,158)(93,146)(94,143)(95,180)$
$(97,169)(99,132)(100,135)(102,121)(104,137)(105,176)(107,188)$
$(112,182)(113,141)(115,134)(116,179)(119,173)(122,127)(123,167)$
$(124,218)(125,152)(126,131)(128,163)(129,144)(130,214)(133,228)$
$(136,140)(138,170)(139,234)(142,237)(145,246)(147,221)(149,251)$
$(151,253)(153,254)(154,244)(155,203)(157,238)(159,191)(162,183)$
$(164,210)(166,264)(171,255)(172,207)(174,201)(175,250)(177,189)$
$(181,242)(184,196)(185,236)(186,241)(187,222)(190,259)(192,194)$
$(193,290)(195,232)(197,212)(198,287)(199,230)(200,209)(202,206)$
$(204,239)(205,285)(208,298)(211,272)(213,279)(215,291)(217,252)$
$(219,275)(220,299)(227,277)(229,283)(231,300)(233,282)(235,296)$
$(240,278)(243,289)(245,265)(247,256)(248,269)(257,273)(258,286)$
$(260,268)(262,294)(263,267)(266,280)(270,297)(271,281)(274,284)$

$tt_2 = (tt_1)^{xx}$
$= (1,5)(2,17)(3,18)(4,15)(6,37)(7,43)(10,34)(11,39)(13,44)(14,40)$
$(16,53)(19,48)(20,70)(21,86)(24,67)(26,74)(29,71)(30,60)(32,81)$
$(33,59)(35,57)(36,94)(38,75)(41,87)(42,80)(45,91)(46,127)(47,138)$
$(49,116)(51,113)(52,125)(55,122)(61,117)(63,129)(64,103)(66,128)$
$(68,108)(69,148)(72,144)(73,105)(76,139)(77,104)(78,123)(79,95)$
$(82,154)(83,194)(84,190)(85,143)(88,175)(89,98)(90,172)(92,196)$
$(93,204)(96,184)(97,102)(99,187)(100,199)(101,192)(106,111)$
$(107,119)(109,197)(110,176)(112,195)(114,167)(115,225)(118,179)$
$(120,212)(121,178)(124,220)(126,216)(130,205)(131,155)(132,185)$
$(133,149)(134,177)(135,280)(136,270)(140,243)(141,163)(142,241)$
$(146,273)(147,278)(150,257)(151,162)(152,158)(153,260)(156,250)$
$(157,159)(160,271)(161,266)(164,267)(165,170)(166,181)(168,274)$
$(171,262)(173,222)(174,293)(180,259)(182,284)(183,226)(186,294)$
$(188,236)(191,249)(198,279)(200,288)(201,248)(202,296)(206,282)$
$(207,232)(208,218)(209,227)(210,258)(211,217)(214,295)(215,255)$
$(219,247)(221,298)(223,287)(224,300)(228,289)(229,245)(230,244)$

$(231,290)(233,272)(234,239)(235,252)(237,291)(240,299)(242,263)$
$(246,281)(251,297)(254,292)(256,261)(264,286)(265,276)$

$tt_3 = (tt_2)^{xx}$
$= (1,4)(2,10)(3,14)(5,12)(6,24)(8,29)(9,33)(11,38)(16,51)(17,31)$
$(18,27)(20,52)(22,45)(23,64)(25,66)(26,63)(28,73)(30,78)(36,90)$
$(37,62)(39,56)(42,99)(43,71)(44,59)(46,83)(48,91)(49,100)(50,110)$
$(53,107)(54,82)(55,109)(57,103)(58,112)(60,102)(61,132)(65,121)$
$(69,142)(70,119)(74,97)(76,79)(77,153)(80,159)(81,105)(84,93)$
$(85,164)(86,128)(87,154)(88,160)(89,169)(92,135)(94,166)(95,145)$
$(96,168)(98,114)(101,120)(104,171)(106,191)(108,176)(111,183)$
$(113,125)(115,208)(116,181)(117,162)(118,210)(122,151)(123,129)$
$(124,133)(126,223)(127,157)(130,131)(134,231)(136,147)(137,233)$
$(138,195)(140,224)(141,238)(143,245)(144,178)(148,235)(149,193)$
$(150,254)(152,253)(155,213)(156,237)(158,188)(161,182)(163,173)$
$(165,265)(170,256)(172,199)(174,257)(175,252)(177,240)(179,247)$
$(180,282)(184,219)(185,197)(186,211)(187,194)(189,289)(190,262)$
$(192,226)(196,229)(198,209)(200,295)(201,281)(202,215)(203,288)$
$(204,260)(206,292)(207,283)(212,249)(216,277)(217,268)(218,300)$
$(225,297)(227,285)(228,298)(230,275)(232,242)(234,296)(239,248)$
$(241,271)(243,251)(244,264)(246,255)(250,269)(258,274)(259,293)$
$(261,266)(267,280)(270,299)(272,273)(276,284)(278,290)$

$tt_0 = (tt_3)^{xx}$
$= (1,2)(3,6)(4,7)(5,8)(9,16)(12,19)(13,20)(14,21)(15,22)(18,25)(23,36)$
$(27,41)(28,42)(31,45)(32,46)(33,47)(34,48)(35,49)(38,52)(39,53)$
$(40,54)(44,58)(50,69)(56,76)(57,77)(59,79)(60,80)(62,82)(63,83)$
$(64,84)(65,85)(67,87)(68,88)(71,91)(72,92)(73,93)(74,94)(75,95)$
$(78,100)(81,104)(89,115)(90,99)(96,101)(97,124)(98,126)(102,130)$
$(103,131)(105,133)(106,118)(107,112)(108,134)(109,135)(110,136)$
$(111,137)(113,139)(114,140)(116,127)(117,143)(119,145)(120,146)$
$(121,147)(122,148)(123,149)(125,138)(128,154)(129,155)(132,160)$

$(141,174)(142,164)(144,177)(150,161)(151,186)(152,189)(153,159)$
$(156,158)(157,193)(162,198)(163,200)(165,180)(166,171)(167,201)$
$(169,202)(170,203)(172,205)(173,206)(175,196)(176,209)(178,211)$
$(181,213)(182,214)(183,215)(184,216)(185,217)(187,220)(188,221)$
$(190,194)(191,224)(192,225)(197,227)(199,204)(207,219)(208,233)$
$(210,223)(212,248)(218,246)(222,261)(226,263)(228,244)(229,268)$
$(230,269)(231,245)(232,272)(234,251)(235,240)(236,275)(238,276)$
$(239,277)(241,279)(242,258)(243,273)(247,281)(249,283)(252,285)$
$(253,286)(255,287)(256,284)(257,288)(264,291)(265,292)(266,293)$
$(267,294)(270,280)(271,278)(274,296)(282,295)$

Thus, tt_0 has exactly four conjugates under conjugation by $\langle xx, yy\rangle$.

(2) $\langle xx, yy\rangle$ acts as S_4 on tt_0, tt_1, tt_2, tt_3 by conjugation,
(That is $xx = (tt_0, tt_1, tt_2, tt_3)$ and $yy = (tt_2, tt_3)$):

We verified above that $xx = (tt_0, tt_1, tt_2, tt_3)$ by the conjugation of tt_i with xx. Now, we must verify that $yy = (tt_2, tt_3)$ by conjugation:
First notice:

$(t_2)^y = t_3$
$(t_2)^{y^2} = t_2$
$\Rightarrow \hat{\alpha}(t_2^y) = \hat{\alpha}(t_2)^{\hat{\alpha}(y)}$

Now we must calculate tt_3 and $(tt_3)^{yy}$:

$(tt_3) = (tt_2)^{yy}$
$\Rightarrow tt_3 = (1,4)(2,10)(3,14)(5,12)(6,24)(8,29)(9,33)(11,38)(16,51)(17,31)$
$(18,27)(20,52)(22,45)(23,64)(25,66)(26,63)(28,73)(30,78)(36,90)$
$(37,62)(39,56)(42,99)(43,71)(44,59)(46,83)(48,91)(49,100)(50,110)$
$(53,107)(54,82)(55,109)(57,103)(58,112)(60,102)(61,132)(65,121)$
$(69,142)(70,119)(74,97)(76,79)(77,153)(80,159)(81,105)(84,93)$
$(85,164)(86,128)(87,154)(88,160)(89,169)(92,135)(94,166)(95,145)$
$(96,168)(98,114)(101,120)(104,171)(106,191)(108,176)(111,183)$
$(113,125)(115,208)(116,181)(117,162)(118,210)(122,151)(123,129)$
$(124,133)(126,223)(127,157)(130,131)(134,231)(136,147)(137,233)$
$(138,195)(140,224)(141,238)(143,245)(144,178)(148,235)(149,193)$
$(150,254)(152,253)(155,213)(156,237)(158,188)(161,182)(163,173)$
$(165,265)(170,256)(172,199)(174,257)(175,252)(177,240)(179,247)$
$(180,282)(184,219)(185,197)(186,211)(187,194)(189,289)(190,262)$

$(192,226)(196,229)(198,209)(200,295)(201,281)(202,215)(203,288)$
$(204,260)(206,292)(207,283)(212,249)(216,277)(217,268)(218,300)$
$(225,297)(227,285)(228,298)(230,275)(232,242)(234,296)(239,248)$
$(241,271)(243,251)(244,264)(246,255)(250,269)(258,274)(259,293)$
$(261,266)(267,280)(270,299)(272,273)(276,284)(278,290)$

$(tt_3)^{yy} = (1,5)(2,17)(3,18)(4,15)(6,37)(7,43)(10,34)(11,39)(13,44)(14,40)$
$(16,53)(19,48)(20,70)(21,86)(24,67)(26,74)(29,71)(30,60)(32,81)$
$(33,59)(35,57)(36,94)(38,75)(41,87)(42,80)(45,91)(46,127)(47,138)$
$(49,116)(51,113)(52,125)(55,122)(61,117)(63,129)(64,103)(66,128)$
$(68,108)(69,148)(72,144)(73,105)(76,139)(77,104)(78,123)(79,95)$
$(82,154)(83,194)(84,190)(85,143)(88,175)(89,98)(90,172)(92,196)$
$(93,204)(96,184)(97,102)(99,187)(100,199)(101,192)(106,111)$
$(107,119)(109,197)(110,176)(112,195)(114,167)(115,225)(118,179)$
$(120,212)(121,178)(124,220)(126,216)(130,205)(131,155)(132,185)$
$(133,149)(134,177)(135,280)(136,270)(140,243)(141,163)(142,241)$
$(146,273)(147,278)(150,257)(151,162)(152,158)(153,260)(156,250)$
$(157,159)(160,271)(161,266)(164,267)(165,170)(166,181)(168,274)$
$(171,262)(173,222)(174,293)(180,259)(182,284)(183,226)(186,294)$
$(188,236)(191,249)(198,279)(200,288)(201,248)(202,296)(206,282)$
$(207,232)(208,218)(209,227)(210,258)(211,217)(214,295)(215,255)$
$(219,247)(221,298)(223,287)(224,300)(228,289)(229,245)(230,244)$
$(231,290)(233,272)(234,239)(235,252)(237,291)(240,299)(242,263)$
$(246,281)(251,297)(254,292)(256,261)(264,286)(265,276)$
$\Rightarrow (tt_3)^{yy} = tt_2$

Therefore $yy = (tt_2, tt_3)$.

Thus, we have shown that $\hat{\alpha}(2^{*4} : S_4) = \hat{\alpha}(N, t_0) =< xx, yy, tt_0 >$ and $< xx, yy, tt_0 >$ is a homomorphic image of $2^{*4} : S_4$. However, we want the homomorphic image of $G = \frac{2^{*4}:S_4}{[(0123)t_0]^{15},[(01)t_0]^5}$. Notice that $< xx, yy, tt_0 >$ is a homomorphic image of G if the additional relations $[(0123)t_0]^{15} = 1$ and $[(01)\,t_0]^5 = 1$ hold in $< xx, yy, tt_0 >$. So, we must verify the following relations: (i) $\hat{\alpha}((01)t_0t_1t_0) = \hat{\alpha}(t_0t_1)$ and (ii) $\hat{\alpha}((0321)t_2t_1t_0t_3t_2t_1t_0t_3) = \hat{\alpha}(t_0t_1t_2t_3t_0t_1t_2)$.

(i) $\hat{\alpha}((01)t_0t_1t_0) = \hat{\alpha}(t_0t_1)$:

On the left hand side of the equation, $\hat{\alpha}((01)t_0t_1t_0) = \hat{\alpha}(01)tt_0tt_1tt_0$
Note: We used magma to compute $\hat{\alpha}(01)$:

$\hat{\alpha}(01) =$

(2,3)(6,11)(7,13)(8,9)(10,14)(16,30)(17,18)(19,35)(20,26)(21,32)(22,23)(24,38)(25,28)
(27,31)(29,33)(34,40)(36,61)(37,39)(41,68)(43,44)(45,64)(47,72)(48,57)(49,55)(50,54)
(51,78)(52,63)(53,60)(56,62)(58,65)(59,71)(66,73)(67,75)(69,118)(70,74)(76,114)
(70,74)(76,114)(77,106)(79,98)(81,86)(82,110)(84,101)(87,108)(88,96)(89,95)(90,132)
(91,103)(93,120)(94,117)(97,119)(100,109)(102,107)(104,111)(105,128)(112,21)
(113,123)(115,180)(116,122)(124,173)(125,129)(126,156)(130,188)(131,158)(133,163)
(134,165)(136,161)(138,144)(139,167)(140,150)(141,149)(142,210)(145,169)(147,182)
(148,179)(151,181)(152,155)(153,191)(154,176)(160,168)(162,166)(170,177)(171,183)
(172,185)(174,251)(175,184)(178,195)(186,242)(189,203)(190,192)(193,238)(197,199)
(198,264)(200,228)(201,234)(202,246)(204,212)(205,236)(206,218)(207,217)(208,282)
(209,244)(211,232)(213,253)(214,221)(215,255)(216,250)(219,252)(220,222)(223,237)
(224,254)(225,259)(226,262)(227,230)(231,265)(235,247)(239,248)(240,256)(241,258)
(243,257)(249,260)(261,299)(263,294)(266,270)(268,283)(269,277)(271,274)(275,285)
(276,290)(278,284)(279,286)(281,296)(287,291)(288,289)(292,300)(293,297)(295,298)
(1,2)(3,6)(4,7)(5,8)(9,16)(12,19)(13,20)(14,21)(15,22)(18,25)(23,36)(27,41)(28,42)
(31,45)(32,46)(33,47)(34,48)(35,49)(38,52)(39,53)(40,54)(44,58)(50,69)(56,76)(57,77)
(59,79)(60,80)(62,82)(63,83)(64,84)(65,85)(67,87)(68,88)(71,91)(72,92)(73,93)(74,94)
(75,95)(78,100)(81,104)(89,115)(90,99)(96,101)(97,124)(98,126)(102,130)(103,131)
(105,133)(106,118)(107,112)(108,134)(109,135)(110,136)(111,137)(113,139)(114,140)
(116,127)(117,143)(119,145)(120,146)(121,147)(122,148)(123,149)(125,138)(128,154)
(129,155)(132,160)(141,174)(142,164)(144,177)(150,161)(151,186)(152,189)(153,159)
(156,158)(157,193)(162,198)(163,200)(165,180)(166,171)(167,201)(169,202)(170,203)
(172,205)(173,206)(175,196)(176,209)(178,211)(181,213)(182,214)(183,215)(184,216)
(185,217)(187,220)(188,221)(190,194)(191,224)(192,225)(197,227)(199,204)(207,219)
(208,233)(210,223)(212,248)(218,246)(222,261)(226,263)(228,244)(229,268)(230,269)
(231,245)(232,272)(234,251)(235,240)(236,275)(238,276)(239,277)(241,279)(242,258)
(243,273)(247,281)(249,283)(252,285)(253,286)(255,287)(256,284)(257,288)(264,291)
(265,292)(266,293)(267,294)(270,280)(271,278)(274,296)(282,295)(1,3)(2,11)(4,13)
(5,9)(7,26)(8,30)(10,32)(12,35)(15,23)(17,28)(1955)(21,46)(22,61)(24,63)(25,42)
(27,64)(29,72)(31,68)(34,50)(37,60)(40,57)(41,96)(43,65)(45,101)(47,92)(48,106)
(51,109)(52,83)(53,80)(54,118)(56,110)(58,85)(59,103)(62,114)(66,120)(67,89)(69,77)

(70, 117)(71, 98)(75, 108)(76, 150)(79, 156)(82, 161)(84, 88)(86, 111)(87, 165)(90, 168)
(91, 158)(93, 146)(94, 143)(95, 180)(97, 169)(99, 132)(100, 135)(102, 121)(104, 137)
(105, 176)(107, 188)(112, 182)(113, 141)(115, 134)(116, 179)(119, 173)(122, 127)(123, 167)
(124, 218)(125, 152)(126, 131)(128, 163)(129, 144)(130, 214)(133, 228)(136, 140)(138, 170)
(139, 234)(142, 237)(145, 246)(147, 221)(149, 251)(151, 253)(153, 254)(154, 244)(155, 203)
(157, 238)(159, 191)(162, 183)(164, 210)(166, 264)(171, 255)(172, 207)(174, 201)(175, 250)
(177, 189)(181, 242)(184, 196)(185, 236)(186, 241)(187, 222)(190, 259)(192, 194)(193, 290)
(195, 232)(197, 212)(198, 287)(199, 230)(200, 209)(202, 206)(204, 239)(205, 285)(208, 298)
(211, 272)(213, 279)(215, 291)(217, 252)(219, 275)(220, 299)(227, 277)(229, 283)(231, 300)
(233, 282)(235, 296)(240, 278)(243, 289)(245, 265)(247, 256)(248, 269)(257, 273)(258, 286)
(260, 268)(262, 294)(263, 267)(266, 280)(270, 297)(271, 281)(274, 284)(1, 2)(3, 6)(4, 7)(5, 8)
(9, 16)(12, 19)(13, 20)(14, 21)(15, 22)(18, 25)(23, 36)(27, 41)(28, 42)(31, 45)(32, 46)(33, 47)
(34, 48)(35, 49)(38, 52)(39, 53)(40, 54)(44, 58)(50, 69)(56, 76)(57, 77)(59, 79)(60, 80)(62, 82)
(63, 83)(64, 84)(65, 85)(67, 87)(68, 88)(71, 91)(72, 92)(73, 93)(74, 94)(75, 95)(78, 100)(81, 104)
(89, 115)(90, 99)(96, 101)(97, 124)(98, 126)(102, 130)(103, 131)(105, 133)(106, 118)(107, 112)
(108, 134)(109, 135)(110, 136)(111, 137)(113, 139)(114, 140)(116, 127)(117, 143)(119, 145)
(120, 146)(121, 147)(122, 148)(123, 149)(125, 138)(128, 154)(129, 155)(132, 160)(141, 174)
(142, 164)(144, 177)(150, 161)(151, 186)(152, 189)(153, 159)(156, 158)(157, 193)(162, 198)
(163, 200)(165, 180)(166, 171)(167, 201)(169, 202)(170, 203)(172, 205)(173, 206)(175, 196)
(176, 209)(178, 211)(181, 213)(182, 214)(183, 215)(184, 216)(185, 217)(187, 220)(188, 221)
(190, 194)(191, 224)(192, 225)(197, 227)(199, 204)(207, 219)(208, 233)(210, 223)(212, 248)
(218, 246)(222, 261)(226, 263)(228, 244)(229, 268)(230, 269)(231, 245)(232, 272)(234, 251)
(235, 240)(236, 275)(238, 276)(239, 277)(241, 279)(242, 258)(243, 273)(247, 281)(249, 283)
(252, 285)(253, 286)(255, 287)(256, 284)(257, 288)(264, 291)(265, 292)(266, 293)(267, 294)
(270, 280)(271, 278)(274, 296)(282, 295)

When we multiply the permutations we get $\hat{\alpha}(01) =$

(1, 11, 2, 3, 6)(4, 26, 7, 13, 20)(5, 30, 8, 9, 16)(10, 32, 21, 14, 46)(12, 55, 19, 35, 49)
(15, 61, 22, 23, 36)(17, 28, 25, 18, 42)(24, 63, 52, 38, 83)(27, 96, 45, 68, 84)(29, 72, 47, 33, 92)
(31, 101, 41, 64, 88)(34, 106, 54, 57, 69)(37, 60, 53, 39, 80)(40, 118, 48, 50, 77)
(43, 65, 58, 44, 85)(51, 109, 100, 78, 135)(56, 150, 82, 114, 136)(59, 156, 91, 98, 131)
(62, 161, 76, 110, 140)(66, 120, 93, 73, 146)(67, 165, 95, 108, 115)(70, 117, 94, 74, 143)
(71, 158, 79, 103, 126)(75, 180, 87, 89, 134)(81, 137, 86, 111, 104)(90, 132, 160, 99, 168)

(97, 218, 145, 173, 202)(102, 214, 112, 188, 147)(105, 228, 154, 163, 209)
(107, 182, 130, 121, 221)(113, 234, 149, 167, 174)(116, 122, 148, 127, 179)
(119, 246, 124, 169, 206)(123, 251, 139, 141, 201)(125, 170, 155, 144, 189)
(128, 244, 133, 176, 200)(129, 203, 138, 152, 177)(142, 210, 223, 164, 237)
(151, 241, 213, 242, 286)(153, 191, 224, 159, 254)(157, 290, 193, 238, 276)
(162, 287, 171, 264, 215)(166, 255, 198, 183, 291)(172, 285, 217, 236, 219)
(175, 184, 216, 196, 250)(178, 272, 195, 232, 211)(181, 279, 186, 253, 258)
(185, 252, 205, 207, 275)(187, 299, 220, 222, 261)(190, 192, 225, 194, 259)
(197, 277, 204, 230, 248)(199, 239, 227, 212, 269)(208, 282, 295, 233, 298)
(226, 267, 262, 294, 263)(229, 260, 268, 283, 249)(231, 265, 292, 245, 300)
(235, 278, 281, 256, 274)(240, 296, 284, 247, 271)(243, 257, 288, 273, 289)
(266, 293, 280, 297, 270)
On the right hand side of the equation,we have $\hat{\alpha}(t_0t_1) = tt_0tt_1 =$
(1, 2)(3, 6)(4, 7)(5, 8)(9, 16)(12, 19)(13, 20)(14, 21)(15, 22)(18, 25)(23, 36)(27, 41)(28, 42)
(31, 45)(32, 46)(33, 47)(34, 48)(35, 49)(38, 52)(39, 53)(40, 54)(44, 58)(50, 69)(56, 76)(57, 77)
(59, 79)(60, 80)(62, 82)(63, 83)(64, 84)(65, 85)(67, 87)(68, 88)(71, 91)(72, 92)(73, 93)(74, 94)
(75, 95)(78, 100)(81, 104)(89, 115)(90, 99)(96, 101)(97, 124)(98, 126)(102, 130)(103, 131)
(105, 133)(106, 118)(107, 112)(108, 134)(109, 135)(110, 136)(111, 137)(113, 139)(114, 140)
(116, 127)(117, 143)(119, 145)(120, 146)(121, 147)(122, 148)(123, 149)(125, 138)(128, 154)
(129, 155)(132, 160)(141, 174)(142, 164)(144, 177)(150, 161)(151, 186)(152, 189)(153, 159)
(156, 158)(157, 193)(162, 198)(163, 200)(165, 180)(166, 171)(167, 201)(169, 202)(170, 203)
(172, 205)(173, 206)(175, 196)(176, 209)(178, 211)(181, 213)(182, 214)(183, 215)(184, 216)
(185, 217)(187, 220)(188, 221)(190, 194)(191, 224)(192, 225)(197, 227)(199, 204)(207, 219)
(208, 233)(210, 223)(212, 248)(218, 246)(222, 261)(226, 263)(228, 244)(229, 268)(230, 269)
(231, 245)(232, 272)(234, 251)(235, 240)(236, 275)(238, 276)(239, 277)(241, 279)(242, 258)
(243, 273)(247, 281)(249, 283)(252, 285)(253, 286)(255, 287)(256, 284)(257, 288)(264, 291)
(265, 292)(266, 293)(267, 294)(270, 280)(271, 278)(274, 296)(282, 295)(1, 3)(2, 11)(4, 13)
(5, 9)(7, 26)(8, 30)(10, 32)(12, 35)(15, 23)(17, 28)(1955)(21, 46)(22, 61)(24, 63)(25, 42)
(27, 64)(29, 72)(31, 68)(34, 50)(37, 60)(40, 57)(41, 96)(43, 65)(45, 101)(47, 92)(48, 106)
(51, 109)(52, 83)(53, 80)(54, 118)(56, 110)(58, 85)(59, 103)(62, 114)(66, 120)(67, 89)(69, 77)
(70, 117)(71, 98)(75, 108)(76, 150)(79, 156)(82, 161)(84, 88)(86, 111)(87, 165)(90, 168)
(91, 158)(93, 146)(94, 143)(95, 180)(97, 169)(99, 132)(100, 135)(102, 121)(104, 137)

$(105,176)(107,188)(112,182)(113,141)(115,134)(116,179)(119,173)(122,127)(123,167)$
$(124,218)(125,152)(126,131)(128,163)(129,144)(130,214)(133,228)(136,140)(138,170)$
$(139,234)(142,237)(145,246)(147,221)(149,251)(151,253)(153,254)(154,244)(155,203)$
$(157,238)(159,191)(162,183)(164,210)(166,264)(171,255)(172,207)(174,201)(175,250)$
$(177,189)(181,242)(184,196)(185,236)(186,241)(187,222)(190,259)(192,194)(193,290)$
$(195,232)(197,212)(198,287)(199,230)(200,209)(202,206)(204,239)(205,285)(208,298)$
$(211,272)(213,279)(215,291)(217,252)(219,275)(220,299)(227,277)(229,283)(231,300)$
$(233,282)(235,296)(240,278)(243,289)(245,265)(247,256)(248,269)(257,273)(258,286)$
$(260,268)(262,294)(263,267)(266,280)(270,297)(271,281)(274,284)$
When we multiply the permutations we get $\hat{\alpha}(t_0t_1) =$
$(1,11,2,3,6)(4,26,7,13,20)(5,30,8,9,16)(10,32,21,14,46)(12,55,19,35,49)$
$(15,61,22,23,36)(17,28,25,18,42)(24,63,52,38,83)(27,96,45,68,84)(29,72,47,33,92)$
$(31,101,41,64,88)(34,106,54,57,69)(37,60,53,39,80)(40,118,48,50,77)$
$(43,65,58,44,85)(51,109,100,78,135)(56,150,82,114,136)(59,156,91,98,131)$
$(62,161,76,110,140)(66,120,93,73,146)(67,165,95,108,115)(70,117,94,74,143)$
$(71,158,79,103,126)(75,180,87,89,134)(81,137,86,111,104)(90,132,160,99,168)$
$(97,218,145,173,202)(102,214,112,188,147)(105,228,154,163,209)$
$(107,182,130,121,221)(113,234,149,167,174)(116,122,148,127,179)$
$(119,246,124,169,206)(123,251,139,141,201)(125,170,155,144,189)$
$(128,244,133,176,200)(129,203,138,152,177)(142,210,223,164,237)$
$(151,241,213,242,286)(153,191,224,159,254)(157,290,193,238,276)$
$(162,287,171,264,215)(166,255,198,183,291)(172,285,217,236,219)$
$(175,184,216,196,250)(178,272,195,232,211)(181,279,186,253,258)$
$(185,252,205,207,275)(187,299,220,222,261)(190,192,225,194,259)$
$(197,277,204,230,248)(199,239,227,212,269)(208,282,295,233,298)$
$(226,267,262,294,263)(229,260,268,283,249)(231,265,292,245,300)$
$(235,278,281,256,274)(240,296,284,247,271)(243,257,288,273,289)$
$(266,293,280,297,270)$
Therefore $\hat{\alpha}((01)t_0t_1t_0) = \hat{\alpha}(t_0t_1)$

(ii) $\hat{\alpha}((0321)t_2t_1t_0t_3t_2t_1t_0t_3) = \hat{\alpha}(t_0t_1t_2t_3t_0t_1t_2)$:
On the left hand side of the equation we have,

$\hat{\alpha}((0321)t_2t_1t_0t_3t_2t_1t_0t_3) = \hat{\alpha}(0321)tt_2tt_1tt_0tt_3tt_2tt_1tt_0tt_3$

Note: We used magma to compute $\hat{\alpha}(0321)$.

$\hat{\alpha}((0321)t_2t_1t_0t_3t_2t_1t_0t_3) =$

$(2,4,5,3)(6,10,15,9)(7,12,18,11)(8,14,17,13)(16,24,34,23)(19,27,39,26)$
$(20,29,40,28)(21,31,44,30)(22,33,37,32)(25,38,43,35)(36,51,67,50)$
$(41,56,74,55)(42,52,71,57)(45,59,60,46)(47,62,81,61)(48,64,53,63)$
$(49,66,75,65)(54,73,70,72)(58,78,86,68)(69,90,113,89)(76,97,122,96)$
$(77,99,125,98)(79,102,127,101)(80,83,91,103)(82,105,117,92)$
$(84,107,129,106)(85,100,128,108)(87,110,94,109)(88,112,123,111)$
$(93,119,144,118)(95,121,116,120)(104,132,138,114)(115,142,172,141)$
$(124,151,184,150)(126,153,187,152)(130,157,192,156)(131,159,194,158)$
$(133,162,196,161)(134,164,199,163)(135,154,176,143)(136,166,197,165)$
$(137,160,195,167)(139,169,148,68)(140,171,185,170)(145,178,179,146)$
$(147,181,212,180)(149,183,175,182)(155,191,190,188)(173,177,210,204)$
$(174,208,241,207)(186,219,257,218)(189,223,260,222)(193,226,250,214)$
$(198,229,266,228)(200,231,267,230)(201,233,271,232)(202,235,274,234)$
$(203,224,262,236)(205,238,225,237)(206,240,258,239)(209,245,280,244)$
$(211,247,273,246)(213,249,259,221)(215,252,284,251)(216,254,220,253)$
$(217,256,243,255)(227,265,270,264)(242,248,282,278)(261,289,287,268)$
$(263,269,295,290)(272,281)(275,288,300,294)(276,297,291,285)$
$(277,292,299,286)(279,283,293,298)(1,5)(2,17)(3,18)(4,15)(6,37)(7,43)(10,34)$
$(11,39)(13,44)(14,40)(16,53)(19,48)(20,70)(21,86)(24,67)(26,74)(29,71)(30,60)$
$(32,81)(33,59)(35,57)(36,94)(38,75)(41,87)(42,80)(45,91)(46,127)(47,138)(49,116)$
$(51,113)(52,125)(55,122)(61,117)(63,129)(64,103)(66,128)(68,108)(69,148)(72,144)$
$(73,105)(76,139)(77,104)(78,123)(79,95)(82,154)(83,194)(84,190)(85,143)(88,175)$
$(89,98)(90,172)(92,196)(93,204)(96,184)(97,102)(99,187)(100,199)(101,192)$
$(106,111)(107,119)(109,197)(110,176)(112,195)(114,167)(115,225)(118,179)$
$(120,212)(121,178)(124,220)(126,216)(130,205)(131,155)(132,185)(133,149)$
$(134,177)(135,280)(136,270)(140,243)(141,163)(142,241)(146,273)(147,278)$
$(150,257)(151,162)(152,158)(153,260)(156,250)(157,159)(160,271)(161,266)$
$(164,267)(165,170)(166,181)(168,274)(171,262)(173,222)(174,293)(180,259)$
$(182,284)(183,226)(186,294)(188,236)(191,249)(198,279)(200,288)(201,248)$

(202, 296)(206, 282)(207, 232)(208, 218)(209, 227)(210, 258)(211, 217)(214, 295)
(215, 255)(219, 247)(221, 298)(223, 287)(224, 300)(228, 289)(229, 245)(230, 244)
(231, 290)(233, 272)(234, 239)(235, 252)(237, 291)(240, 299)(242, 263)(246, 281)
(251, 297)(254, 292)(256, 261)(264, 286)(265, 276)(1, 3)(2, 11)(4, 13)(5, 9)(7, 26)
(8, 30)(10, 32)(12, 35)(15, 23)(17, 28)(1955)(21, 46)(22, 61)(24, 63)(25, 42)(27, 64)
(29, 72)(31, 68)(34, 50)(37, 60)(40, 57)(41, 96)(43, 65)(45, 101)(47, 92)(48, 106)(51, 109)
(52, 83)(53, 80)(54, 118)(56, 110)(58, 85)(59, 103)(62, 114)(66, 120)(67, 89)(69, 77)
(70, 117)(71, 98)(75, 108)(76, 150)(79, 156)(82, 161)(84, 88)(86, 111)(87, 165)(90, 168)
(91, 158)(93, 146)(94, 143)(95, 180)(97, 169)(99, 132)(100, 135)(102, 121)(104, 137)
(105, 176)(107, 188)(112, 182)(113, 141)(115, 134)(116, 179)(119, 173)(122, 127)
(123, 167)(124, 218)(125, 152)(126, 131)(128, 163)(129, 144)(130, 214)(133, 228)
(136, 140)(138, 170)(139, 234)(142, 237)(145, 246)(147, 221)(149, 251)(151, 253)
(153, 254)(154, 244)(155, 203)(157, 238)(159, 191)(162, 183)(164, 210)(166, 264)
(171, 255)(172, 207)(174, 201)(175, 250)(177, 189)(181, 242)(184, 196)(185, 236)
(186, 241)(187, 222)(190, 259)(192, 194)(193, 290)(195, 232)(197, 212)(198, 287)
(199, 230)(200, 209)(202, 206)(204, 239)(205, 285)(208, 298)(211, 272)(213, 279)
(215, 291)(217, 252)(219, 275)(220, 299)(227, 277)(229, 283)(231, 300)(233, 282)
(235, 296)(240, 278)(243, 289)(245, 265)(247, 256)(248, 269)(257, 273)(258, 286)
(260, 268)(262, 294)(263, 267)(266, 280)(270, 297)(271, 281)(274, 284)(1, 2)(3, 6)
(4, 7)(5, 8)(9, 16)(12, 19)(13, 20)(14, 21)(15, 22)(18, 25)(23, 36)(27, 41)(28, 42)(31, 45)
(32, 46)(33, 47)(34, 48)(35, 49)(38, 52)(39, 53)(40, 54)(44, 58)(50, 69)(56, 76)(57, 77)
(59, 79)(60, 80)(62, 82)(63, 83)(64, 84)(65, 85)(67, 87)(68, 88)(71, 91)(72, 92)(73, 93)
(74, 94)(75, 95)(78, 100)(81, 104)(89, 115)(90, 99)(96, 101)(97, 124)(98, 126)(102, 130)
(103, 131)(105, 133)(106, 118)(107, 112)(108, 134)(109, 135)(110, 136)(111, 137)
(113, 139)(114, 140)(116, 127)(117, 143)(119, 145)(120, 146)(121, 147)(122, 148)
(123, 149)(125, 138)(128, 154)(129, 155)(132, 160)(141, 174)(142, 164)(144, 177)
(150, 161)(151, 186)(152, 189)(153, 159)(156, 158)(157, 193)(162, 198)(163, 200)
(165, 180)(166, 171)(167, 201)(169, 202)(170, 203)(172, 205)(173, 206)(175, 196)
(176, 209)(178, 211)(181, 213)(182, 214)(183, 215)(184, 216)(185, 217)(187, 220)
(188, 221)(190, 194)(191, 224)(192, 225)(197, 227)(199, 204)(207, 219)(208, 233)
(210, 223)(212, 248)(218, 246)(222, 261)(226, 263)(228, 244)(229, 268)(230, 269)
(231, 245)(232, 272)(234, 251)(235, 240)(236, 275)(238, 276)(239, 277)(241, 279)

(242, 258)(243, 273)(247, 281)(249, 283)(252, 285)(253, 286)(255, 287)(256, 284)
(257, 288)(264, 291)(265, 292)(266, 293)(267, 294)(270, 280)(271, 278)(274, 296)
(282, 295)(1, 4)(2, 10)(3, 14)(5, 12)(6, 24)(8, 29)(9, 33)(11, 38)(16, 51)(17, 31)(18, 27)
(20, 52)(22, 45)(23, 64)(25, 66)(26, 63)(28, 73)(30, 78)(36, 90)(37, 62)(39, 56)(42, 99)
(43, 71)(44, 59)(46, 83)(48, 91)(49, 100)(50, 110)(53, 107)(54, 82)(55, 109)(57, 103)
(58, 112)(60, 102)(61, 132)(65, 121)(69, 142)(70, 119)(74, 97)(76, 79)(77, 153)(80, 159)
(81, 105)(84, 93)(85, 164)(86, 128)(87, 154)(88, 160)(89, 169)(92, 135)(94, 166)(95, 145)
(96, 168)(98, 114)(101, 120)(104, 171)(106, 191)(108, 176)(111, 183)(113, 125)(115, 208)
(116, 181)(117, 162)(118, 210)(122, 151)(123, 129)(124, 133)(126, 223)(127, 157)
(130, 131)(134, 231)(136, 147)(137, 233)(138, 195)(140, 224)(141, 238)(143, 245)
(144, 178)(148, 235)(149, 193)(150, 254)(152, 253)(155, 213)(156, 237)(158, 188)
(161, 182)(163, 173)(165, 265)(170, 256)(172, 199)(174, 257)(175, 252)(177, 240)
(179, 247)(180, 282)(184, 219)(185, 197)(186, 211)(187, 194)(189, 289)(190, 262)
(192, 226)(196, 229)(198, 209)(200, 295)(201, 281)(202, 215)(203, 288)(204, 260)
(206, 292)(207, 283)(212, 249)(216, 277)(217, 268)(218, 300)(225, 297)(227, 285)
(228, 298)(230, 275)(232, 242)(234, 296)(239, 248)(241, 271)(243, 251)(244, 264)
(246, 255)(250, 269)(258, 274)(259, 293)(261, 266)(267, 280)(270, 299)(272, 273)
(276, 284)(278, 290)(1, 5)(2, 17)(3, 18)(4, 15)(6, 37)(7, 43)(10, 34)(11, 39)(13, 44)
(14, 40)(16, 53)(19, 48)(20, 70)(21, 86)(24, 67)(26, 74)(29, 71)(30, 60)(32, 81)(33, 59)
(35, 57)(36, 94)(38, 75)(41, 87)(42, 80)(45, 91)(46, 127)(47, 138)(49, 116)(51, 113)
(52, 125)(55, 122)(61, 117)(63, 129)(64, 103)(66, 128)(68, 108)(69, 148)(72, 144)
(73, 105)(76, 139)(77, 104)(78, 123)(79, 95)(82, 154)(83, 194)(84, 190)(85, 143)
(88, 175)(89, 98)(90, 172)(92, 196)(93, 204)(96, 184)(97, 102)(99, 187)(100, 199)
(101, 192)(106, 111)(107, 119)(109, 197)(110, 176)(112, 195)(114, 167)(115, 225)
(118, 179)(120, 212)(121, 178)(124, 220)(126, 216)(130, 205)(131, 155)(132, 185)
(133, 149)(134, 177)(135, 280)(136, 270)(140, 243)(141, 163)(142, 241)(146, 273)
(147, 278)(150, 257)(151, 162)(152, 158)(153, 260)(156, 250)(157, 159)(160, 271)
(161, 266)(164, 267)(165, 170)(166, 181)(168, 274)(171, 262)(173, 222)(174, 293)
(180, 259)(182, 284)(183, 226)(186, 294)(188, 236)(191, 249)(198, 279)(200, 288)
(201, 248)(202, 296)(206, 282)(207, 232)(208, 218)(209, 227)(210, 258)(211, 217)
(214, 295)(215, 255)(219, 247)(221, 298)(223, 287)(224, 300)(228, 289)(229, 245)
(230, 244)(231, 290)(233, 272)(234, 239)(235, 252)(237, 291)(240, 299)(242, 263)

(246, 281)(251, 297)(254, 292)(256, 261)(264, 286)(265, 276)(1, 3)(2, 11)(4, 13)(5, 9)
(7, 26)(8, 30)(10, 32)(12, 35)(15, 23)(17, 28)(1955)(21, 46)(22, 61)(24, 63)(25, 42)
(27, 64)(29, 72)(31, 68)(34, 50)(37, 60)(40, 57)(41, 96)(43, 65)(45, 101)(47, 92)(48, 106)
(51, 109)(52, 83)(53, 80)(54, 118)(56, 110)(58, 85)(59, 103)(62, 114)(66, 120)(67, 89)
(69, 77)(70, 117)(71, 98)(75, 108)(76, 150)(79, 156)(82, 161)(84, 88)(86, 111)(87, 165)
(90, 168)(91, 158)(93, 146)(94, 143)(95, 180)(97, 169)(99, 132)(100, 135)(102, 121)
(104, 137)(105, 176)(107, 188)(112, 182)(113, 141)(115, 134)(116, 179)(119, 173)
(122, 127)(123, 167)(124, 218)(125, 152)(126, 131)(128, 163)(129, 144)(130, 214)
(133, 228)(136, 140)(138, 170)(139, 234)(142, 237)(145, 246)(147, 221)(149, 251)
(151, 253)(153, 254)(154, 244)(155, 203)(157, 238)(159, 191)(162, 183)(164, 210)
(166, 264)(171, 255)(172, 207)(174, 201)(175, 250)(177, 189)(181, 242)(184, 196)
(185, 236)(186, 241)(187, 222)(190, 259)(192, 194)(193, 290)(195, 232)(197, 212)
(198, 287)(199, 230)(200, 209)(202, 206)(204, 239)(205, 285)(208, 298)(211, 272)
(213, 279)(215, 291)(217, 252)(219, 275)(220, 299)(227, 277)(229, 283)(231, 300)
(233, 282)(235, 296)(240, 278)(243, 289)(245, 265)(247, 256)(248, 269)(257, 273)
(258, 286)(260, 268)(262, 294)(263, 267)(266, 280)(270, 297)(271, 281)(274, 284)
(1, 2)(3, 6)(4, 7)(5, 8)(9, 16)(12, 19)(13, 20)(14, 21)(15, 22)(18, 25)(23, 36)(27, 41)
(28, 42)(31, 45)(32, 46)(33, 47)(34, 48)(35, 49)(38, 52)(39, 53)(40, 54)(44, 58)(50, 69)
(56, 76)(57, 77)(59, 79)(60, 80)(62, 82)(63, 83)(64, 84)(65, 85)(67, 87)(68, 88)(71, 91)
(72, 92)(73, 93)(74, 94)(75, 95)(78, 100)(81, 104)(89, 115)(90, 99)(96, 101)(97, 124)
(98, 126)(102, 130)(103, 131)(105, 133)(106, 118)(107, 112)(108, 134)(109, 135)
(110, 136)(111, 137)(113, 139)(114, 140)(116, 127)(117, 143)(119, 145)(120, 146)
(121, 147)(122, 148)(123, 149)(125, 138)(128, 154)(129, 155)(132, 160)(141, 174)
(142, 164)(144, 177)(150, 161)(151, 186)(152, 189)(153, 159)(156, 158)(157, 193)
(162, 198)(163, 200)(165, 180)(166, 171)(167, 201)(169, 202)(170, 203)(172, 205)
(173, 206)(175, 196)(176, 209)(178, 211)(181, 213)(182, 214)(183, 215)(184, 216)
(185, 217)(187, 220)(188, 221)(190, 194)(191, 224)(192, 225)(197, 227)(199, 204)
(207, 219)(208, 233)(210, 223)(212, 248)(218, 246)(222, 261)(226, 263)(228, 244)
(229, 268)(230, 269)(231, 245)(232, 272)(234, 251)(235, 240)(236, 275)(238, 276)
(239, 277)(241, 279)(242, 258)(243, 273)(247, 281)(249, 283)(252, 285)(253, 286)
(255, 287)(256, 284)(257, 288)(264, 291)(265, 292)(266, 293)(267, 294)(270, 280)
(271, 278)(274, 296)(282, 295)(1, 4)(2, 10)(3, 14)(5, 12)(6, 24)(8, 29)(9, 33)(11, 38)

(16, 51)(17, 31)(18, 27)(20, 52)(22, 45)(23, 64)(25, 66)(26, 63)(28, 73)(30, 78)(36, 90)
(37, 62)(39, 56)(42, 99)(43, 71)(44, 59)(46, 83)(48, 91)(49, 100)(50, 110)(53, 107)
(54, 82)(55, 109)(57, 103)(58, 112)(60, 102)(61, 132)(65, 121)(69, 142)(70, 119)(74, 97)
(76, 79)(77, 153)(80, 159)(81, 105)(84, 93)(85, 164)(86, 128)(87, 154)(88, 160)(89, 169)
(92, 135)(94, 166)(95, 145)(96, 168)(98, 114)(101, 120)(104, 171)(106, 191)(108, 176)
(111, 183)(113, 125)(115, 208)(116, 181)(117, 162)(118, 210)(122, 151)(123, 129)
(124, 133)(126, 223)(127, 157)(130, 131)(134, 231)(136, 147)(137, 233)(138, 195)
(140, 224)(141, 238)(143, 245)(144, 178)(148, 235)(149, 193)(150, 254)(152, 253)
(155, 213)(156, 237)(158, 188)(161, 182)(163, 173)(165, 265)(170, 256)(172, 199)
(174, 257)(175, 252)(177, 240)(179, 247)(180, 282)(184, 219)(185, 197)(186, 211)
(187, 194)(189, 289)(190, 262)(192, 226)(196, 229)(198, 209)(200, 295)(201, 281)
(202, 215)(203, 288)(204, 260)(206, 292)(207, 283)(212, 249)(216, 277)(217, 268)
(218, 300)(225, 297)(227, 285)(228, 298)(230, 275)(232, 242)(234, 296)(239, 248)
(241, 271)(243, 251)(244, 264)(246, 255)(250, 269)(258, 274)(259, 293)(261, 266)
(267, 280)(270, 299)(272, 273)(276, 284)(278, 290)

When we multiply the permutations, we get: $\hat{\alpha}((0321)t_2t_1t_0t_3t_2t_1t_0t_3) =$
(1, 257, 272, 13, 281, 174)(2, 184, 267, 32, 230, 261)(3, 266, 200)(4, 208, 229, 14, 231, 219)
(5, 295, 228, 35, 198, 115)(6, 122, 258, 26, 152, 276)(7, 250, 187, 63, 239, 226)
(8, 196, 241, 72, 207, 201)(9, 284, 126)(10, 289, 157, 38, 240, 151)(11, 192, 206)
(12, 142, 252, 33, 153, 269)(15, 223, 251, 64, 130, 189)(16, 74, 291, 55, 89, 202)
(17, 273, 218, 68, 186, 134)(18, 292, 156, 23, 215, 69)(19, 216, 113, 109, 285, 253)
(20, 117, 185, 46, 111, 249)(21, 179, 123, 83, 170, 178)(22, 175, 172, 120, 268, 275)
(24, 297, 97)(25, 127, 287, 101, 141, 209)(27, 90, 277)(28, 212, 88)
(29, 233, 119, 78, 171, 162)(30, 144, 140)(31, 164, 181, 73, 112, 247)
(34, 237, 299, 110, 76, 225)(36, 39, 274, 96, 50, 286)(37, 280, 150, 98, 263, 77)
(40, 160, 180, 103, 93, 137)(41, 148, 67, 168, 234, 169)(42, 81, 221, 61, 108, 136)
(43, 298, 290, 114, 124, 177)(44, 224, 118, 57, 147, 85)(45, 288, 128, 132, 213, 183)
(47, 143, 75, 135, 165, 176)(48, 243, 260, 188, 193, 248)(49, 60, 197, 92, 65, 236)
(51, 235, 56)(52, 259, 105)(53, 194, 214, 158, 222, 131)(54, 116, 199, 182, 232, 283)
(58, 70, 271, 161, 163, 211)(59, 100, 203)(62, 99, 166, 102, 66, 245)
(71, 210, 262, 121, 82, 293)(79, 94, 125, 254, 244, 238)(80, 129, 146, 106, 167, 84)
(86, 300, 246, 173, 133, 104)(87, 270, 205, 264, 220, 227)(91, 282, 195, 191, 145, 256)

$(95,190,204,255,294,217)(107,159)(138,278,279,242,149,155)(139,296)(154,265)$

On the right hand side of the equation, $\hat{\alpha}(t_0t_1t_2t_3t_0t_1t_2) = tt_0tt_1tt_2tt_3tt_0tt_1tt_2$
Hence $\hat{\alpha}(t_0t_1t_2t_3t_0t_1t_2) =$

$(1,2)(3,6)(4,7)(5,8)(9,16)(12,19)(13,20)(14,21)(15,22)(18,25)(23,36)(27,41)$
$(28,42)(31,45)(32,46)(33,47)(34,48)(35,49)(38,52)(39,53)(40,54)(44,58)(50,69)$
$(56,76)(57,77)(59,79)(60,80)(62,82)(63,83)(64,84)(65,85)(67,87)(68,88)(71,91)$
$(72,92)(73,93)(74,94)(75,95)(78,100)(81,104)(89,115)(90,99)(96,101)(97,124)$
$(98,126)(102,130)(103,131)(105,133)(106,118)(107,112)(108,134)(109,135)(110,136)$
$(111,137)(113,139)(114,140)(116,127)(117,143)(119,145)(120,146)(121,147)(122,148)$
$(123,149)(125,138)(128,154)(129,155)(132,160)(141,174)(142,164)(144,177)(150,161)$
$(151,186)(152,189)(153,159)(156,158)(157,193)(162,198)(163,200)(165,180)(166,171)$
$(167,201)(169,202)(170,203)(172,205)(173,206)(175,196)(176,209)(178,211)(181,213)$
$(182,214)(183,215)(184,216)(185,217)(187,220)(188,221)(190,194)(191,224)(192,225)$
$(197,227)(199,204)(207,219)(208,233)(210,223)(212,248)(218,246)(222,261)(226,263)$
$(228,244)(229,268)(230,269)(231,245)(232,272)(234,251)(235,240)(236,275)(238,276)$
$(239,277)(241,279)(242,258)(243,273)(247,281)(249,283)(252,285)(253,286)(255,287)$
$(256,284)(257,288)(264,291)(265,292)(266,293)(267,294)(270,280)(271,278)(274,296)$
$(282,295)(1,3)(2,11)(4,13)(5,9)(7,26)(8,30)(10,32)(12,35)(15,23)(17,28)(1955)(21,46)$
$(22,61)(24,63)(25,42)(27,64)(29,72)(31,68)(34,50)(37,60)(40,57)(41,96)(43,65)$
$(45,101)(47,92)(48,106)(51,109)(52,83)(53,80)(54,118)(56,110)(58,85)(59,103)$
$(62,114)(66,120)(67,89)(69,77)(70,117)(71,98)(75,108)(76,150)(79,156)(82,161)$
$(84,88)(86,111)(87,165)(90,168)(91,158)(93,146)(94,143)(95,180)(97,169)(99,132)$
$(100,135)(102,121)(104,137)(105,176)(107,188)(112,182)(113,141)(115,134)(116,179)$
$(119,173)(122,127)(123,167)(124,218)(125,152)(126,131)(128,163)(129,144)(130,214)$
$(133,228)(136,140)(138,170)(139,234)(142,237)(145,246)(147,221)(149,251)(151,253)$
$(153,254)(154,244)(155,203)(157,238)(159,191)(162,183)(164,210)(166,264)(171,255)$
$(172,207)(174,201)(175,250)(177,189)(181,242)(184,196)(185,236)(186,241)(187,222)$
$(190,259)(192,194)(193,290)(195,232)(197,212)(198,287)(199,230)(200,209)(202,206)$
$(204,239)(205,285)(208,298)(211,272)(213,279)(215,291)(217,252)(219,275)$
$(220,299)(227,277)(229,283)(231,300)(233,282)(235,296)(240,278)(243,289)(245,265)$

(247, 256)(248, 269)(257, 273)(258, 286)(260, 268)(262, 294)(263, 267)(266, 280)
(270, 297)(271, 281)(274, 284)(1, 5)(2, 17)(3, 18)(4, 15)(6, 37)(7, 43)(10, 34)(11, 39)
(13, 44)(14, 40)(16, 53)(19, 48)(20, 70)(21, 86)(24, 67)(26, 74)(29, 71)(30, 60)(32, 81)
(33, 59)(35, 57)(36, 94)(38, 75)(41, 87)(42, 80)(45, 91)(46, 127)(47, 138)(49, 116)(51, 113)
(52, 125)(55, 122)(61, 117)(63, 129)(64, 103)(66, 128)(68, 108)(69, 148)(72, 144)(73, 105)
(76, 139)(77, 104)(78, 123)(79, 95)(82, 154)(83, 194)(84, 190)(85, 143)(88, 175)(89, 98)
(90, 172)(92, 196)(93, 204)(96, 184)(97, 102)(99, 187)(100, 199)(101, 192)(106, 111)
(107, 119)(109, 197)(110, 176)(112, 195)(114, 167)(115, 225)(118, 179)(120, 212)
(121, 178)(124, 220)(126, 216)(130, 205)(131, 155)(132, 185)(133, 149)(134, 177)
(135, 280)(136, 270)(140, 243)(141, 163)(142, 241)(146, 273)(147, 278)(150, 257)
(151, 162)(152, 158)(153, 260)(156, 250)(157, 159)(160, 271)(161, 266)(164, 267)
(165, 170)(166, 181)(168, 274)(171, 262)(173, 222)(174, 293)(180, 259)(182, 284)
(183, 226)(186, 294)(188, 236)(191, 249)(198, 279)(200, 288)(201, 248)(202, 296)
(206, 282)(207, 232)(208, 218)(209, 227)(210, 258)(211, 217)(214, 295)(215, 255)
(219, 247)(221, 298)(223, 287)(224, 300)(228, 289)(229, 245)(230, 244)(231, 290)
(233, 272)(234, 239)(235, 252)(237, 291)(240, 299)(242, 263)(246, 281)(251, 297)
(254, 292)(256, 261)(264, 286)(265, 276)(1, 4)(2, 10)(3, 14)(5, 12)(6, 24)(8, 29)(9, 33)
(11, 38)(16, 51)(17, 31)(18, 27)(20, 52)(22, 45)(23, 64)(25, 66)(26, 63)(28, 73)(30, 78)
(36, 90)(37, 62)(39, 56)(42, 99)(43, 71)(44, 59)(46, 83)(48, 91)(49, 100)(50, 110)(53, 107)
(54, 82)(55, 109)(57, 103)(58, 112)(60, 102)(61, 132)(65, 121)(69, 142)(70, 119)(74, 97)
(76, 79)(77, 153)(80, 159)(81, 105)(84, 93)(85, 164)(86, 128)(87, 154)(88, 160)(89, 169)
(92, 135)(94, 166)(95, 145)(96, 168)(98, 114)(101, 120)(104, 171)(106, 191)(108, 176)
(111, 183)(113, 125)(115, 208)(116, 181)(117, 162)(118, 210)(122, 151)(123, 129)
(124, 133)(126, 223)(127, 157)(130, 131)(134, 231)(136, 147)(137, 233)(138, 195)
(140, 224)(141, 238)(143, 245)(144, 178)(148, 235)(149, 193)(150, 254)(152, 253)
(155, 213)(156, 237)(158, 188)(161, 182)(163, 173)(165, 265)(170, 256)(172, 199)
(174, 257)(175, 252)(177, 240)(179, 247)(180, 282)(184, 219)(185, 197)(186, 211)
(187, 194)(189, 289)(190, 262)(192, 226)(196, 229)(198, 209)(200, 295)(201, 281)
(202, 215)(203, 288)(204, 260)(206, 292)(207, 283)(212, 249)(216, 277)(217, 268)
(218, 300)(225, 297)(227, 285)(228, 298)(230, 275)(232, 242)(234, 296)(239, 248)
(241, 271)(243, 251)(244, 264)(246, 255)(250, 269)(258, 274)(259, 293)(261, 266)
(267, 280)(270, 299)(272, 273)(276, 284)(278, 290)(1, 2)(3, 6)(4, 7)(5, 8)(9, 16)(12, 19)

(13, 20)(14, 21)(15, 22)(18, 25)(23, 36)(27, 41)(28, 42)(31, 45)(32, 46)(33, 47)(34, 48)
(35, 49)(38, 52)(39, 53)(40, 54)(44, 58)(50, 69)(56, 76)(57, 77)(59, 79)(60, 80)(62, 82)
(63, 83)(64, 84)(65, 85)(67, 87)(68, 88)(71, 91)(72, 92)(73, 93)(74, 94)(75, 95)(78, 100)
(81, 104)(89, 115)(90, 99)(96, 101)(97, 124)(98, 126)(102, 130)(103, 131)(105, 133)
(106, 118)(107, 112)(108, 134)(109, 135)(110, 136)(111, 137)(113, 139)(114, 140)
(116, 127)(117, 143)(119, 145)(120, 146)(121, 147)(122, 148)(123, 149)(125, 138)
(128, 154)(129, 155)(132, 160)(141, 174)(142, 164)(144, 177)(150, 161)(151, 186)
(152, 189)(153, 159)(156, 158)(157, 193)(162, 198)(163, 200)(165, 180)(166, 171)
(167, 201)(169, 202)(170, 203)(172, 205)(173, 206)(175, 196)(176, 209)(178, 211)
(181, 213)(182, 214)(183, 215)(184, 216)(185, 217)(187, 220)(188, 221)(190, 194)
(191, 224)(192, 225)(197, 227)(199, 204)(207, 219)(208, 233)(210, 223)(212, 248)
(218, 246)(222, 261)(226, 263)(228, 244)(229, 268)(230, 269)(231, 245)(232, 272)
(234, 251)(235, 240)(236, 275)(238, 276)(239, 277)(241, 279)(242, 258)(243, 273)
(247, 281)(249, 283)(252, 285)(253, 286)(255, 287)(256, 284)(257, 288)(264, 291)
(265, 292)(266, 293)(267, 294)(270, 280)(271, 278)(274, 296)(282, 295)(1, 3)(2, 11)
(4, 13)(5, 9)(7, 26)(8, 30)(10, 32)(12, 35)(15, 23)(17, 28)(1955)(21, 46)(22, 61)(24, 63)
(25, 42)(27, 64)(29, 72)(31, 68)(34, 50)(37, 60)(40, 57)(41, 96)(43, 65)(45, 101)(47, 92)
(48, 106)(51, 109)(52, 83)(53, 80)(54, 118)(56, 110)(58, 85)(59, 103)(62, 114)(66, 120)
(67, 89)(69, 77)(70, 117)(71, 98)(75, 108)(76, 150)(79, 156)(82, 161)(84, 88)(86, 111)
(87, 165)(90, 168)(91, 158)(93, 146)(94, 143)(95, 180)(97, 169)(99, 132)(100, 135)
(102, 121)(104, 137)(105, 176)(107, 188)(112, 182)(113, 141)(115, 134)(116, 179)
(119, 173)(122, 127)(123, 167)(124, 218)(125, 152)(126, 131)(128, 163)(129, 144)
(130, 214)(133, 228)(136, 140)(138, 170)(139, 234)(142, 237)(145, 246)(147, 221)
(149, 251)(151, 253)(153, 254)(154, 244)(155, 203)(157, 238)(159, 191)(162, 183)
(164, 210)(166, 264)(171, 255)(172, 207)(174, 201)(175, 250)(177, 189)(181, 242)
(184, 196)(185, 236)(186, 241)(187, 222)(190, 259)(192, 194)(193, 290)(195, 232)
(197, 212)(198, 287)(199, 230)(200, 209)(202, 206)(204, 239)(205, 285)(208, 298)
(211, 272)(213, 279)(215, 291)(217, 252)(219, 275)(220, 299)(227, 277)(229, 283)
(231, 300)(233, 282)(235, 296)(240, 278)(243, 289)(245, 265)(247, 256)(248, 269)
(257, 273)(258, 286)(260, 268)(262, 294)(263, 267)(266, 280)(270, 297)(271, 281)
(274, 284)(1, 5)(2, 17)(3, 18)(4, 15)(6, 37)(7, 43)(10, 34)(11, 39)(13, 44)(14, 40)
(16, 53)(19, 48)(20, 70)(21, 86)(24, 67)(26, 74)(29, 71)(30, 60)(32, 81)(33, 59)(35, 57)

(36, 94)(38, 75)(41, 87)(42, 80)(45, 91)(46, 127)(47, 138)(49, 116)(51, 113)(52, 125)
(55, 122)(61, 117)(63, 129)(64, 103)(66, 128)(68, 108)(69, 148)(72, 144)(73, 105)(76, 139)
(77, 104)(78, 123)(79, 95)(82, 154)(83, 194)(84, 190)(85, 143)(88, 175)(89, 98)(90, 172)
(92, 196)(93, 204)(96, 184)(97, 102)(99, 187)(100, 199)(101, 192)(106, 111)(107, 119)
(109, 197)(110, 176)(112, 195)(114, 167)(115, 225)(118, 179)(120, 212)(121, 178)
(124, 220)(126, 216)(130, 205)(131, 155)(132, 185)(133, 149)(134, 177)(135, 280)
(136, 270)(140, 243)(141, 163)(142, 241)(146, 273)(147, 278)(150, 257)(151, 162)
(152, 158)(153, 260)(156, 250)(157, 159)(160, 271)(161, 266)(164, 267)(165, 170)
(166, 181)(168, 274)(171, 262)(173, 222)(174, 293)(180, 259)(182, 284)(183, 226)
(186, 294)(188, 236)(191, 249)(198, 279)(200, 288)(201, 248)(202, 296)(206, 282)
(207, 232)(208, 218)(209, 227)(210, 258)(211, 217)(214, 295)(215, 255)(219, 247)
(221, 298)(223, 287)(224, 300)(228, 289)(229, 245)(230, 244)(231, 290)(233, 272)
(234, 239)(235, 252)(237, 291)(240, 299)(242, 263)(246, 281)(251, 297)(254, 292)
(256, 261)(264, 286)(265, 276)

When we multiply the permutation, we get:

(1, 257, 272, 13, 281, 174)(2, 184, 267, 32, 230, 261)(3, 266, 200)(4, 208, 229, 14, 231, 219)
(5, 295, 228, 35, 198, 115)(6, 122, 258, 26, 152, 276)(7, 250, 187, 63, 239, 226)
(8, 196, 241, 72, 207, 201)(9, 284, 126)(10, 289, 157, 38, 240, 151)(11, 192, 206)
(12, 142, 252, 33, 153, 269)(15, 223, 251, 64, 130, 189)(16, 74, 291, 55, 89, 202)
(17, 273, 218, 68, 186, 134)(18, 292, 156, 23, 215, 69)(19, 216, 113, 109, 285, 253)
(20, 117, 185, 46, 111, 249)(21, 179, 123, 83, 170, 178)(22, 175, 172, 120, 268, 275)
(24, 297, 97)(25, 127, 287, 101, 141, 209)(27, 90, 277)(28, 212, 88)(29, 233, 119, 78, 171, 162)
(30, 144, 140)(31, 164, 181, 73, 112, 247)(34, 237, 299, 110, 76, 225)(36, 39, 274, 96, 50, 286)
(37, 280, 150, 98, 263, 77)(40, 160, 180, 103, 93, 137)(41, 148, 67, 168, 234, 169)
(42, 81, 221, 61, 108, 136)(43, 298, 290, 114, 124, 177)(44, 224, 118, 57, 147, 85)
(45, 288, 128, 132, 213, 183)(47, 143, 75, 135, 165, 176)(48, 243, 260, 188, 193, 248)
(49, 60, 197, 92, 65, 236)(51, 235, 56)(52, 259, 105)(53, 194, 214, 158, 222, 131)
(54, 116, 199, 182, 232, 283)(58, 70, 271, 161, 163, 211)(59, 100, 203)(62, 99, 166, 102, 66, 245)
(71, 210, 262, 121, 82, 293)(79, 94, 125, 254, 244, 238)(80, 129, 146, 106, 167, 84)
(86, 300, 246, 173, 133, 104)(87, 270, 205, 264, 220, 227)(91, 282, 195, 191, 145, 256)
(95, 190, 204, 255, 294, 217)(107, 159)(138, 278, 279, 242, 149, 155)(139, 296)(154, 265)

Therefore $\hat{\alpha}((0321)t_2t_1t_0t_3t_2t_1t_0t_3) = \hat{\alpha}\,(t_0t_1t_2t_3t_0t_1t_2)$

By the First Isomorphism Theorem, we have that

$\frac{G}{ker\hat{\alpha}} \cong \hat{\alpha}(G)$

$\Rightarrow \frac{|G|}{|ker\hat{\alpha}|} \cong |\langle xx, yy, tt_0\rangle| = 7200$

$\Rightarrow |G| = 7200 \cdot |ker\hat{\alpha}|$

$\Rightarrow |G| \geq 7200$

But, using the cayley diagram, we had previously shown that $|G| \leq 7200$. Therefore $|G| = 7200$.

4.3 Permutation and Symmetric Representation

Permutation to Symmetric: The goal is to find the symmetric representation form of the following permutation p.

Let $p =$

(1, 105, 284, 175, 205, 96, 38, 169, 152, 37, 113, 221, 210, 242, 106)(2, 97, 215, 216, 139, 156, 24, 253, 230, 53, 128, 140, 164, 217, 165)(3, 59, 212, 196, 204, 150, 63, 110, 89, 39, 67, 291, 237, 264, 158)(4, 178, 266, 116, 260, 92, 78, 238, 111, 17, 187, 224, 282, 294, 61)(5, 123, 292, 243, 279, 55, 14, 176, 192, 70, 172, 146132, 173, 65)(6, 56, 286, 250, 87, 228, 52, 183, 207, 80, 71, 206, 142, 227, 234)(7, 151, 198, 148, 154, 126, 51, 275, 239, 25, 119, 136, 233, 149, 180)(8, 157, (8, 157, 147, 288, 220, 118, 10, 226, 281, 94,195, 88, 99, 283, 170)(9, 40, 197, 273, 278, 161, 32, 64, 108, 74,125, 287, 168, 188, 98)(11, 27, 236, 184, 138, 218, 83, 121, 141, 60, 34, 258, 223, 296, 244)(12, 249, 257, 190, 290, 46, 73, 222, 117, 43, 267, 160, 265, 232, 30)(13, 103, 144, 127, 199, 214, 109, 114, 50, 18, 75, 274, 298, 255, 101)(15, 185, 295, 270, 246, 26, 33,167, 122, 86, 241, 135, 191, 163, 28)(16, 62, 202, 289, 95, 251, 21, 162, 186, 143, 91, 200, 90, 248, 285)(19, 219, 130, 225, 145, 69, 66, 201, 240, 58, 181, 84, 245, 133, 203)(20, 102, 263, 179, 48, 272, 100, 256, 268, 42, 31, 276, 208, 155, 299)(22, 229, 93, 293,193, 85, 29, 261, 231, 104, 262, 49, 159, 211, 259)(23, 44, 129, 280, 300, 182, 72, 35, 57, 81, 194, 271, 254, 120, 68)(36, 107, 209, 297, 54, 252, 47, 247, 124, 137, 45, 174, 153, 177, 213)(41, 269, 76, 189, 82, 115, 112, 134, 171, 77, 166, 131, 235, 79, 277)$\in S_{300}$

By definition, the right cosets of any group, partition the group into disjoint subsets. Hence, any two right cosets of a group will either be equal or disjoint.

Since $Np = Nw$ and $p \in Np$

$\Rightarrow p \in Nw$

$\Rightarrow p = nw$, where $n \in N$ and $N \cong S_4$

In order to find the symmetric representation form of p, we must solve for this n.

$\Rightarrow n = pw^{-1}$.

But $Np = 1^p = 105$ and , from the labeling, $105 = Nt_4t_2t_1t_3t_2$

$\Rightarrow Np = Nt_4t_2t_1t_3t_2$

$\Rightarrow p = nt_4t_2t_1t_3t_2$

$\Rightarrow n = pt_2t_3t_1t_2t_4$

Notice that $pt_2t_3t_1t_2t_4 = (2,3,7)(4,6)(8,21,20)(9,18,22,17,11,19)(10,12,16)$
$(13,15)(23,26,27,32,29,24)(28,31,30)$

Therefore, $n = (2,3)$. Now we must look at the labeling of the double cosets, and translate the permutation $(2,3)$:

$\Rightarrow n = (4,1)$

Thus, $(4,1)t_4t_2t_1t_3t_2$ is the symmetric representation form of the permutation p.

Chapter 5

The Construction of $A_7 \times S_5$

We want to prove that the symmetric representations of $A_7 \times S_5$ is given by $\frac{3^{*5}:_m S_5}{[(01)(23)t_0]^4}$, where $X \sim (01234)\left(\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\right)$ and $Y \sim (0\bar{0})\,(1\bar{1})\,(2\bar{2})\,(3\bar{4})\,(4\bar{3})$. The symmetric presentation of the progenitor $3^{*5} :_m S_5$ is :

$3^{*5} :_m S_5 \cong \left\langle x, y, t | x^5 = y^2 = (yx)^4 = [x,y]^3 = 1 = t^3 = (yt)^2 = (yt^x)^2, \left(yt^{x^2}\right)^2 \right\rangle$.

The control subgroup N is S_5 on ten letters, t_i and $\bar{t_i}$ for $i = 0, .., 4$.

To prove:

$A_7 \times S_5 \cong \left\langle x, y, t | x^5 = y^2 = (yx)^4 = [x,y]^3 = 1 = t^3 = (yt)^2 = (yt^x)^2, \left(yt^{x^2}\right)^2 \right\rangle$,

we must perform a double coset enumeration of $A_7 \times S_5$ over S_5. The double coset enumeration consists of finding all double cosets $[w]$ and figuring out how many single cosets each of these contains. Notice that the double coset enumeration is complete when the set of right cosets is closed under right multiplication. That is, when none of the t_i's can expand the cayley graph any further. From the definition of 3^{*5} we obtain the following relation: $t_0^3 = t_1^3 = t_2^3 = t_3^3 = t_4^3 = 1$. Since $t_i^3 = 1$ for $i = 0, ..., 4$, then $t_i^3 \cdot t_i^{-1} = 1 \cdot t_i^{-1} \Longrightarrow t_i^2 = t_i^{-1}$. Thus we obtain this new relation $t_i^2 = t_i^{-1}$. The control subgroup is permutation group S_5 on ten letters which consists of the permutations of $0, 1, 2, 3$ and 4, as well as $\bar{0}, \bar{1}, \bar{2}, \bar{3}$ and $\bar{4}$. That is, $S_5 = \langle (01234)(\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}), (0\bar{0})(1\bar{1})(2\bar{2})(3\bar{4})(4\bar{3}) \rangle$

We will proceed to expand the relators and obtain our relations.

The first relation is as follows: $[(01)\,(23)\,t_0]^4 = e$

Let $\pi = (01)\,(23)$ .Then by permutation multiplication, we obtain the following:

$\pi^1 = (01)\,(23) = \pi^3$

$\pi^2 = e = \pi^4$.

So if we expand $[(01)(23)t_0]^4 = e$ we obtain the following:

$\pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$

Using the property that $\pi^n \pi^{-n} = e$ we can simplify the expression

$\Rightarrow \pi\pi^3\pi^{-3}t_0\pi\pi^2\pi^{-2}t_0\pi\pi^1\pi^{-1}t_0\pi t_0 = e$

$\Rightarrow \pi^4\left(\pi^{-3}t_0\pi^3\right)\left(\pi^{-2}t\pi^2\right)\left(\pi^{-1}t\pi^1\right)t_0 = e$

Using the conjugation property, $\pi^{-1}t_i\pi = t_i^{\pi}$,

$\Rightarrow \pi^4 t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi^1} t_0 = e$

$\Rightarrow t_0^{(01)(23)} t_0^{e} t_0^{(01)(23)^1} t_0 = e$

$\Rightarrow t_1 t_0 t_1 t_0 = e$

$\Rightarrow t_1 t_0 = t_0 t_1$

Thus, we obtain the relation $t_1 t_0 = t_0 t_1$

The second relation is as follows: $[(0\bar{1})(2\bar{3}4\bar{2}3\bar{4})t_0]^6 = e$.

Let $\pi = (0\bar{1})(2\bar{3}4\bar{2}3\bar{4})$. Then, by permutation multiplication, we obtain the following

$\pi^1 = (0\bar{1})(2\bar{3}4\bar{2}3\bar{4})$

$\pi^2 = (243)(\bar{2}\bar{4}\bar{3})$

$\pi^3 = (0\bar{1})(2\bar{2})(3\bar{3})(4\bar{4})$

$\pi^4 = (234)(\bar{2}\bar{3}\bar{4})$

$\pi^5 = (0\bar{1})(2\bar{4}3\bar{2}4\bar{3})$

$\pi^6 = e$

So, if we expand $[(0\bar{1})(\bar{0}1)(2\bar{3}4\bar{2}3\bar{4})t_0]^6 = e$ we obtain the following

$\pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 \pi t_0 = e$

$\Rightarrow \pi^6 t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi^1} t_0 = e$

$\Rightarrow t_0^{(0\bar{1})(2\bar{4}3\bar{2}4\bar{3})} t_0^{(234)(\bar{2}\bar{3}\bar{4})} t_0^{(0\bar{1})(2\bar{2})(3\bar{3})(4\bar{4})} t_0^{(243)(\bar{2}\bar{4}\bar{3})} t_0^{(0\bar{1})(2\bar{3}4\bar{2}3\bar{4})} t_0 = e$

$\Rightarrow \bar{t_1} t_0 \bar{t_1} t_0 \bar{t_1} t_0 = e$

$\Rightarrow \bar{t_1} t_0 \bar{t_1} = t_0 \bar{t_1} t_0$

Thus, we obtain the relation $\bar{t_1} t_0 \bar{t_1} = t_0 \bar{t_1} t_0$.

The third relation is as follows $[(0\bar{1})(2\bar{3}4\bar{2}3\bar{4})\bar{t_0}]^6 = e$

Let $\pi = (0\bar{1})(2\bar{3}4\bar{2}3\bar{4})$. Then, by permutation multiplication, we obtain the following

$\pi^1 = (0\bar{1})(\bar{0}1)(2\bar{3}4\bar{2}3\bar{4})$

$\pi^2 = (243)(\bar{2}\bar{4}\bar{3})$

$\pi^3 = (0\bar{1})(\bar{0}1)(2\bar{2})(3\bar{3})(4\bar{4})$

$\pi^4 = (234)(\bar{2}\bar{3}\bar{4})$

$\pi^5 = (0\bar{1})\,(\bar{0}1)\,(2\bar{4}3\bar{2}4\bar{3})$

$\pi^6 = e$

So, if we expand $[(0\bar{1})\,(2\bar{3}4\bar{2}3\bar{4})\,\bar{t}_0]^6 = e$ we obtain the following

$\pi\bar{t}_0\pi\bar{t}_0\pi\bar{t}_0\pi\bar{t}_0\pi\bar{t}_0\pi\bar{t}_0 = e$

$\Rightarrow \pi^6\bar{t}_0^{\pi^5}\bar{t}_0^{\pi^4}\bar{t}_0^{\pi^3}\bar{t}_0^{\pi^2}\bar{t}_0^{\pi^1}\bar{t}_0 = e$

$\Rightarrow \bar{t}_0^{(0\bar{1})(\bar{0}1)(2\bar{4}3\bar{2}4\bar{3})}\bar{t}_0^{(234)(\bar{2}\bar{3}\bar{4})}\bar{t}_0^{(0\bar{1})(\bar{0}1)(2\bar{2})(3\bar{3})(4\bar{4})}\bar{t}_0^{(243)(\bar{2}\bar{4}\bar{3})}\bar{t}_0^{(0\bar{1})(\bar{0}1)(2\bar{3}4\bar{2}3\bar{4})}\bar{t}_0 = e$

$\Rightarrow t_1\bar{t}_0t_1\bar{t}_0t_1\bar{t}_0 = e$

$\Rightarrow t_1\bar{t}_0t_1 = t_0\bar{t}_1t_0$

Thus, we obtain the relation $t_1\bar{t}_0t_1 = t_0\bar{t}_1t_0$.

From relations 2 and 3 we obtain the following relation:
By transitiviy, since $\bar{t}_1t_0\bar{t}_1 = t_0\bar{t}_1t_0$ and $t_1\bar{t}_0t_1 = t_0\bar{t}_1t_0$, then $\bar{t}_1t_0\bar{t}_1 = t_1\bar{t}_0t_1$
Thus we obtain a new relation $\bar{t}_1t_0\bar{t}_1 = t_1\bar{t}_0t_1$.

5.1 The Double Coset Enumeration of G over S_5

First, note that $N = \langle (0,\bar{0})\,(1,\bar{1})\,(2,\bar{2})\,(3,\bar{4})\,(4,\bar{3}), (0,1,2,3,4)\,(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})\rangle$. The empty word is $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$. We will denote this double coset by $[*]$. The number of elements in $[*]$ is $\frac{|N|}{|N|} = \frac{120}{120} = 1$, hence $[*]$ consists of the single coset, N. Since $N \cong S_5$,it acts transitively on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Therefore, N has a single orbit, namely $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ in $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Take a representative from this orbit and apply it to N, and determine to which double coset it belongs to (in other words, if the Cayley graph is expanded):

$N \cdot t_0 = Nt_0 \in [0]$.
Since the orbit of 0 is of length ten,
ten t_i's will extend the Cayley graph from $[*]$ to $[0]$.

Next, look at the double coset Nt_0N , the words of length one, which we will denote by [0]. To obtain the elements in [0], we must first find the point stabiliser of 0, N^0. The point stabiliser,N^0, consists of permutations in $N = S_5$ which fix 0 and permute $1,2,3,4$. This means that $|N^0| = 4! = 24$. In this case, the point stabiliser is equal to the set stabiliser, $N^{(0)}$. $N^{(0)} = N^0 = \langle (1,2,3,4), (\bar{1},\bar{2},\bar{3},\bar{4})\rangle = \langle (1,2,3)\,(1,4), (\bar{1},\bar{2},\bar{3})\,(\bar{1},\bar{4})\rangle$

$= \langle (1,2)(1,3)(1,4), (\bar{1},\bar{2})(\bar{1},\bar{3})(\bar{1},\bar{4}) \rangle \cong A_4$ on 8 letters. Thus, $\left|N^{(0)}\right| = \frac{4!}{2} = 12$. Hence To find the orbits of $N^{(0)}$, we conjugate $0, 1, 2, 3$ and 4 by $N^{(0)}$. The orbits of $N^{(0)}$ are $\{0\}, \{\bar{0}\}, \{1,2,3,4\}$ and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$. The number of single cosets in the double coset $[0]$ is obtained by dividing $\frac{|N|}{\left|N^{(0)}\right|} = \frac{120}{12} = 10$. To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(0)}$ in N. Therefore $[0] = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}, N\bar{t_3}, N\bar{t_4}\}$.

Now we must take a representative from each orbit of $N^{(0)}$ and apply it to Nt_0 to determine if the Cayley graph is expanded or collapsed:

$Nt_0 \cdot t_0 = N\bar{t_0} \in [0]$.
Since the orbit of 0 is of length 1,
one t_i will map $[0]$ to $[0]$.

$Nt_0 \cdot \bar{t_0} = N \in [*]$
Since the orbit of $\bar{0}$ is of length 1,
one t_i will collapse the Cayley graph from $[0]$ to $[*]$.

$Nt_0 \cdot t_1 = Nt_0t_1$
This is a new double coset which we will label as $[01]$.
Since the orbit of 1 is of length 4,
four t_i's will extend the Cayley graph from $[0]$ to $[01]$.

$Nt_0 \cdot \bar{t_1} = Nt_0\bar{t_1}$
This is a new double coset which we will label as $[0\bar{1}]$.
Since the orbit of 1 is of length 4,
four t_i's will extend the Cayley graph from $[0]$ to $[0\bar{1}]$.

Now, we look at the double coset Nt_0t_1N which we denoted $[01]$. To obtain the elements in $[01]$, we must first find the point stabiliser of 0 and 1, N^{01}. The point stabiliser consists of permutations in $N = S_5$ which fix 0 and 1 and permute $2, 3, 4$. This means that $\left|N^{01}\right| = 3! = 6$. The set stabiliser, $N^{(01)} \geq N^{01}$. Due to relation 1,

listed above, the set stabiliser increases:

$$N\left(t_0t_1\right)^{(01)} = Nt_1t_0 = Nt_0t_1 \Longrightarrow (01) \in N^{(01)} \Longrightarrow N^{(01)} \geq \left\langle N^{01}, (01)\right\rangle.$$

Therefore, $\left|N^{(01)}\right| = 6 \cdot 2 = 12$. To find the orbits of $N^{(01)}$, we conjugate $0, 1, 2, 3$ and 4 by $N^{(01)}$. The orbits of $N^{(01)}$ are $\{0, 1\}$, $\{2, 3, 4\}$, and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. The number of single cosets in the double coset $[01]$ is obtained by dividing $\frac{|N|}{\left|N^{(01)}\right|} = \frac{120}{12} = 10$. To obtain the single cosets in this double coset, we apply the permutations in S_5 :

$[01] = Nt_0t_1N = \{N\left(t_0t_1\right)^n | n \in N\} = \{Nt_0t_1, Nt_0t_2, Nt_0t_3, Nt_0t_4\}$

Next, we must take a representative of each orbit of $N^{(01)}$ and apply it to Nt_0t_1 to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1 \cdot t_1 = Nt_0\left(t_1\right)^2 = Nt_0\bar{t}_1 \in [0\bar{1}]$.

Since the orbit of 1 is of length 2,

2 things map from $[01]$ to $[0\bar{1}]$.

$Nt_0t_1 \cdot t_2 = Nt_0t_1t_2$.

This is a new double coset which we will label as $[012]$.

Since the orbit of 2 is of length 3,

three things will extend the Cayley graph from $[01]$ to $[012]$.

$Nt_0t_1 \cdot \bar{t}_1 = Nt_0 \in [0]$.

Since the orbit of $\bar{1}$ is of length 5, five things map from $[01]$ to $[0]$.

Now we loook at the double coset $Nt_0\bar{t}_1N$ which we denoted $[0\bar{1}]$. To obtain the elements in $[0\bar{1}]$, we must first find the point stabiliser of 0 and $\bar{1}$, $N^{0\bar{1}}$. The point stabiliser, $N^{0\bar{1}}$, consists of permutations in $N = S_5$ which fix 0 and $\bar{1}$ permute $2, 3, 4$. Hence $N^{0\bar{1}} = \langle(234)\left(\bar{2}\bar{3}\bar{4}\right)\rangle$. The set stabiliser $N^{(0\bar{1})} = N^{0\bar{1}}$, since there are no additionnal relations which increase it, which means $\left|N^{(0\bar{1})}\right| = 3$. To find the orbits of $N^{(0\bar{1})}$, we conjugate $0, 1, 2, 3$ and 4 by $N^{(0\bar{1})}$. The orbits of $N^{(0\bar{1})}$ are $\{0\}, \{1\}, \{2, 3, 4\}, \{\bar{0}\}, \{\bar{1}\}$, and $\{\bar{2}, \bar{3}, \bar{4}\}$. The number of single cosets in the double coset $[0\bar{1}]$ is obtained by dividing $\frac{|N|}{\left|N^{(0\bar{1})}\right|} = \frac{120}{3} = 40$. To obtain the single cosets in this double coset:

$[0\bar{1}] = Nt_0\bar{t}_1N = \{N\left(t_0\bar{t}_1\right)^n | n \in N\} =$

$\{Nt_0\bar{t}_1, Nt_1\bar{t}_2, N\bar{t}_0t_1, Nt_2\bar{t}_3, N\bar{t}_1t_2, Nt_3\bar{t}_4, N\bar{t}_2t_4, N\bar{t}_2t_3, Nt_4\bar{t}_0, N\bar{t}_4t_3, N\bar{t}_3t_0, N\bar{t}_3t_4,$

$Nt_2\bar{t}_4, N\bar{t}_0t_4, N\bar{t}_4t_1, N\bar{t}_4t_0, Nt_4\bar{t}_3, Nt_3\bar{t}_0, N\bar{t}_1t_0, Nt_0\bar{t}_3, N\bar{t}_0t_2, Nt_3\bar{t}_1, Nt_0\bar{t}_4, Nt_4\bar{t}_1,$
$\mathrm{N}_2t_1, Nt_1\bar{t}_0, Nt_1\bar{t}_4, N\bar{t}_1t_3, Nt_0\bar{t}_2, Nt_4\bar{t}_2, N\bar{t}_0t_3, N\bar{t}_3t_1, N\bar{t}_3t_2, Nt_2\bar{t}_1, Nt_2\bar{t}_0, Nt_1\bar{t}_3,$
$N\bar{t}_1t_4, N\bar{t}_4t_2, Nt_3\bar{t}_2, N\bar{t}_2t_0\}$

Next, we must take a representative of each orbit of $N^{(0\bar{1})}$ and apply it to $Nt_0\bar{t}_1$ to determine if it will extend or collapse the Cayley graph:

$Nt_0\bar{t}_1 \cdot t_0 = Nt_0\bar{t}_1t_0$.
This is a new double coset which we will label as $[0\bar{1}0]$.
Since the orbit of 0 is of length 1,
one t_i will extend the Cayley graph from $[0\bar{1}]$ to $[0\bar{1}0]$.

$Nt_0\bar{t}_1 \cdot t_1 = Nt_0 \in [0]$.
Since the orbit of 1 is of length 1,
one t_i will collapse the Cayley graph from $[0\bar{1}]$ to $[0]$.

$Nt_0\bar{t}_1 \cdot t_2 = Nt_0\bar{t}_1t_2$.
This is a new double coset which we will label as $[0\bar{1}2]$.
Since the orbit of 2 is of length 3,
three t_i's will extend the Cayley graph from $[0\bar{1}]$ to $[0\bar{1}2]$.

$Nt_0\bar{t}_1 \cdot \bar{t}_0 = Nt_0\bar{t}_1\bar{t}_0 \in [01]$.
Since the orbit of $\bar{0}$ is of length 1;
one t_i will map the Cayley graph from $[0\bar{1}]$ to $[0\bar{1}]$.

$Nt_0\bar{t}_1 \cdot \bar{t}_1 = Nt_0t_1 \in [01]$.
Since the orbit of 1 is of length 1,
one t_i will map the Cayley graph from $[0\bar{1}]$ to $[01]$.

$Nt_0\bar{t}_1 \cdot \bar{t}_2 = Nt_0\bar{t}_1\bar{t}_2 =$.
Since the orbit of $\bar{2}$ is of length 3,

three t_i's will map the Cayley graph from $[0\bar{1}]$ to $[012]$.

Now we look at the double coset $Nt_0t_1t_2N$ which we denoted by $[012]$. To obtain the elements in $[012]$ we must first find the point stabiliser of $0, 1$, and 2, N^{012}. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2$ and permute $3, 4$, hence $\left|N^{012}\right| = 1$. Since there are no additional relations which would increase these permutations, we get that the set stabiliser $N^{(012)} = N^{012}$. Therefore, $\left|N^{(012)}\right| = 1$. To find the orbits of $N^{(012)}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(012)}$. The orbits of $N^{(012)}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[012]$ is obtained by dividing $\frac{|N|}{\left|N^{(012)}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset:

$[012] = Nt_0t_1t_2N = \{N(t_0t_1t_2)^n \,|n \in N\} =$
$\{Nt_0t_1t_2, Nt_3t_1t_2, Nt_4t_1t_2, Nt_0t_1t_3, Nt_4t_1t_3, Nt_2t_1t_3, Nt_2t_1t_0, Nt_4t_1t_0, Nt_3t_1t_0, Nt_0t_1t_4,$
$Nt_2t_1t_4, Nt_3t_1t_4, Nt_1t_2t_3, Nt_4t_2t_3, Nt_0t_2t_3, Nt_1t_2t_4, Nt_0t_2t_4, Nt_3t_2t_4, Nt_3t_2t_1, Nt_0t_2t_1,$
$Nt_4t_2t_1, Nt_1t_2t_0, Nt_3t_2t_0, Nt_4t_2t_0, N\bar{t}_0\bar{t}_1\bar{t}_2, N\bar{t}_4\bar{t}_1\bar{t}_2, N\bar{t}_3\bar{t}_1\bar{t}_2, N\bar{t}_0\bar{t}_1\bar{t}_4, N\bar{t}_3\bar{t}_1\bar{t}_4, N\bar{t}_2\bar{t}_1\bar{t}_4,$
$N\bar{t}_2\bar{t}_1\bar{t}_0, N\bar{t}_3\bar{t}_1\bar{t}_0, N\bar{t}_4\bar{t}_1\bar{t}_0, N\bar{t}_0\bar{t}_1\bar{t}_3, N\bar{t}_2\bar{t}_1\bar{t}_3, N\bar{t}_4\bar{t}_1\bar{t}_3, Nt_4t_0t_1, Nt_2t_0t_1, Nt_3t_0t_1, Nt_4t_0t_2,$
$Nt_3t_0t_2, Nt_1t_0t_2, Nt_1t_0t_4, Nt_3t_0t_4, Nt_2t_0t_4, Nt_4t_0t_3, Nt_1t_0t_3, Nt_2t_0t_3, Nt_2t_3t_4, Nt_0t_3t_4,$
$Nt_1t_3t_4, Nt_2t_3t_0, Nt_1t_3t_0, Nt_4t_3t_0, Nt_4t_3t_2, Nt_1t_3t_2, Nt_0t_3t_2, Nt_2t_3t_1, Nt_4t_3t_1, Nt_0t_3t_1,$
$N\bar{t}_1\bar{t}_2\bar{t}_4, N\bar{t}_3\bar{t}_2\bar{t}_4, N\bar{t}_0\bar{t}_2\bar{t}_4, N\bar{t}_1\bar{t}_2\bar{t}_3, N\bar{t}_0\bar{t}_2\bar{t}_3, N\bar{t}_4\bar{t}_2\bar{t}_3, N\bar{t}_4\bar{t}_2\bar{t}_1, N\bar{t}_0\bar{t}_2\bar{t}_1, N\bar{t}_3\bar{t}_2\bar{t}_1, N\bar{t}_1\bar{t}_2\bar{t}_0,$
$N\bar{t}_4\bar{t}_2\bar{t}_0, N\bar{t}_3\bar{t}_2\bar{t}_0, N\bar{t}_4\bar{t}_0\bar{t}_1, N\bar{t}_3\bar{t}_0\bar{t}_1, N\bar{t}_2\bar{t}_0\bar{t}_1, N\bar{t}_4\bar{t}_0\bar{t}_3, N\bar{t}_2\bar{t}_0\bar{t}_3, N\bar{t}_1\bar{t}_0\bar{t}_3, N\bar{t}_1\bar{t}_0\bar{t}_4, N\bar{t}_2\bar{t}_0\bar{t}_4,$
$N\bar{t}_3\bar{t}_0\bar{t}_4, N\bar{t}_4\bar{t}_0\bar{t}_2, N\bar{t}_1\bar{t}_0\bar{t}_2, N\bar{t}_3\bar{t}_0\bar{t}_2, Nt_3t_4t_0, Nt_1t_4t_0, Nt_2t_4t_0, Nt_3t_4t_1, Nt_2t_4t_1, Nt_0t_4t_1,$
$Nt_0t_4t_3, Nt_2t_4t_3, Nt_1t_4t_3, Nt_3t_4t_2, Nt_0t_4t_2, Nt_1t_4t_2, N\bar{t}_2\bar{t}_4\bar{t}_3, N\bar{t}_0\bar{t}_4\bar{t}_3, N\bar{t}_1\bar{t}_4\bar{t}_3, N\bar{t}_2\bar{t}_4\bar{t}_0,$
$N\bar{t}_1\bar{t}_4\bar{t}_0, N\bar{t}_3\bar{t}_4\bar{t}_1, N\bar{t}_3\bar{t}_4\bar{t}_2, N\bar{t}_1\bar{t}_4\bar{t}_2, N\bar{t}_0\bar{t}_4\bar{t}_2, N\bar{t}_2\bar{t}_4\bar{t}_1, N\bar{t}_3\bar{t}_4\bar{t}_1, N\bar{t}_0\bar{t}_4\bar{t}_1, N\bar{t}_2\bar{t}_3\bar{t}_0, N\bar{t}_4\bar{t}_3\bar{t}_0,$
$N\bar{t}_1\bar{t}_3\bar{t}_0, N\bar{t}_2\bar{t}_3\bar{t}_4, N\bar{t}_1\bar{t}_3\bar{t}_1, N\bar{t}_0\bar{t}_3\bar{t}_1, N\bar{t}_0\bar{t}_3\bar{t}_2, N\bar{t}_1\bar{t}_3\bar{t}_2, N\bar{t}_4\bar{t}_3\bar{t}_2, N\bar{t}_2\bar{t}_3\bar{t}_1, N\bar{t}_0\bar{t}_3\bar{t}_1, N\bar{t}_4\bar{t}_3\bar{t}_1\}$

Next, we must take a representative of each orbit of $N^{(012)}$ and apply it to $Nt_0t_1t_2$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2 \cdot t_0 = Nt_0t_1t_2t_0 = Nt_0\bar{t}_1t_2.$
Since the orbit of 0 is of length 1,
one t_i will map from $[012]$ to $[0\bar{1}2]$.

$Nt_0t_1t_2 \cdot t_1 = Nt_0t_1t_2t_1 = Nt_0\bar{t}_1 \in [0\bar{1}]$.
Since the orbit of 1 is of length 1,
one t_i will map from [012] to $[0\bar{1}]$.

$Nt_0t_1t_2 \cdot t_2 = Nt_0t_1t_2t_2 = Nt_0t_1t_2 \in [012]$.
Since the orbit of 2 is of length 1,
one t_i will map from [012] to [012].

$Nt_0t_1t_2 \cdot t_3 = Nt_0t_1t_2t_3$.
This is a new double coset which we will label as [0123].
Since the orbit of 3 is of length 1,
one t_i will extend the Cayley graph from [012] to [0123].

$Nt_0t_1t_2 \cdot t_4 = Nt_0t_1t_2t_4 = Nt_0t_1t_2t_3 \in [0123]$.
Since the orbit of 4 is of length 1,
one t_i will extend the Cayley graph from [012] to [0123].

$Nt_0t_1t_2 \cdot \bar{t}_0 = Nt_0t_1t_2\bar{t}_0$.
This is a new double coset which we will label as $[012\bar{0}]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will extend the Cayley graph from [012] to $[012\bar{0}]$.

$Nt_0t_1t_2 \cdot \bar{t}_1 = Nt_0t_1t_2\bar{t}_1 \in [0\bar{1}2]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the Cayley graph from [012] to $[0\bar{1}2]$.

$Nt_0t_1t_2 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_2 = Nt_0t_1 \in [01]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will collapse the Cayley graph from [012] to [01].

$Nt_0t_1t_2 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_3$.
This is a new double coset which we will label as $[012\bar{3}]$.

Since the orbit of $\bar{3}$ is of length one,
one t_i will expand the Cayley graph from [012] to [012$\bar{3}$].

$Nt_0t_1t_2 \cdot \bar{t}_4 = Nt_0t_1t_2\bar{t}_4$.
This is a new double coset which we will label as [012$\bar{4}$].
Since the orbit of $\bar{4}$ is of length one,
one t_i will expand the Cayley graph from [012] to [012$\bar{4}$].

Now, we look at the double coset $Nt_0\bar{t}_1t_0$ which we denoted [0$\bar{1}$0]. To obtain the elements in [0$\bar{1}$0] we must first find the point stabiliser of 0 and $\bar{1}$, $N^{0\bar{1}0}$. The point stabiliser consists of permutations in $N = S_5$ which fix $0, \bar{1}$, and permute $2, 3, 4$. Hence $N^{0\bar{1}0} = S_3$ which means $\left|N^{0\bar{1}0}\right| = 3! = 6$. By relation 3, we know that $\bar{t}_1t_0\bar{t}_1 = t_0\bar{t}_1t_0$. Notice that:
$Nt_0\bar{t}_1t_0\,(01) = N\,(t_0\bar{t}_1t_0)^{(01)} = Nt_1\bar{t}_0t_1 = Nt_0\bar{t}_1t_0$, by relation 3.
$\Longrightarrow$ Thus, $(01) \in N^{0\bar{1}0}$.
$\Longrightarrow N^{(0\bar{1}0)} \geq \left\langle N^{0\bar{1}0}, (01)\right\rangle$.
$\Longrightarrow \left|N^{(0\bar{1}0)}\right| = 12$.
To find the orbits of $N^{(0\bar{1}0)}$, we must congujate $0, 1, 2, 3$, and 4 by $N^{(0\bar{1}0)}$. The orbits are $\{0, 1, \bar{0}, \bar{1}\}$ and $\{2, 3, 4, \bar{2}, \bar{3}, \bar{4}\}$. The number of single cosets in the double coset [0$\bar{1}$0] is obtained by dividing $\frac{|N|}{\left|N^{(0\bar{1}0)}\right|} = \frac{120}{12} = 10$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[0\bar{1}0] = Nt_0\bar{t}_1t_0 = \{N\,(t_0\bar{t}_1t_0)^n \,|n \in N\} =$
$\{Nt_0\bar{t}_1t_0, Nt_1\bar{t}_2t_1, Nt_2\bar{t}_3t_2, Nt_3\bar{t}_4t_3, Nt_4\bar{t}_0t_4, Nt_0\bar{t}_3t_0, Nt_4\bar{t}_1t_4, N\bar{t}_0t_2\bar{t}_0, N\bar{t}_2t_4\bar{t}_2, N\bar{t}_1t_3\bar{t}_1\}$

Next, we must take a representative from each orbit of $N^{(0\bar{1}0)}$ and apply it to $Nt_0\bar{t}_1t_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0\bar{t}_1t_0 \cdot \bar{t}_0 = Nt_0\bar{t}_1 \in [0\bar{1}]$.
Since the orbit of $\bar{0}$ is of length 4,
four t_i's will collapse the Cayley graph from [0$\bar{1}$0] to [0$\bar{1}$]

$Nt_0\bar{t}_1t_0 \cdot t_2 = Nt_0\bar{t}_1t_0t_2 = Nt_0t_1t_2\bar{t}_0 \in [012\bar{0}]$.

Since the orbit of 2 is of length 6,
six t_i's will extend the Cayley graph from $[0\bar{1}0]$ to $[012\bar{0}]$.

Now, we look at the double coset $Nt_0\bar{t}_1t_2N$ which we denoted $[0\bar{1}2]$. To obtain the elements in $[0\bar{1}2]$ we must first find the point stabiliser of $0, \bar{1}$ and 2, $N^{0\bar{1}2}$. The point stabiliser consists of permutations in $N = S_5$ which fix $0, \bar{1}, 2$ and permute $2, 3, 4$. Hence $\left|N^{0\bar{1}2}\right| = 1$. Since there are no relations that would increase the stabiliser, we get that $N^{(0\bar{1}2)} = N^{0\bar{1}2}$, hence $\left|N^{(0\bar{1}2)}\right| = 1$. To find the orbits of $N^{(0\bar{1}2)}$, we must congujate $0, 1, 2, 3$, and 4 by $N^{(0\bar{1}2)}$. The orbits are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[0\bar{1}2]$ is obtained by dividing $\frac{|N|}{|N^{(0\bar{1}2)}|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[0\bar{1}2] = Nt_0\bar{t}_1t_2 = \{N(t_0\bar{t}_1t_2)^n \,|n \in N\} =$

$\{Nt_0\bar{t}_1t_2, Nt_3\bar{t}_1t_2, Nt_4\bar{t}_1t_2, Nt_5\bar{t}_1t_3, Nt_4\bar{t}_1t_3, Nt_2\bar{t}_1t_3, Nt_2\bar{t}_1t_0, Nt_4\bar{t}_1t_0, Nt_3\bar{t}_1t_0, Nt_0\bar{t}_1t_4$
$Nt_2\bar{t}_1t_4, Nt_3\bar{t}_1t_4, Nt_1\bar{t}_2t_3, Nt_4\bar{t}_2t_3, Nt_0\bar{t}_2t_3, Nt_1\bar{t}_2t_4, Nt_0\bar{t}_2t_4, Nt_3\bar{t}_2t_4, Nt_3\bar{t}_2t_1, Nt_0\bar{t}_2t_1$
$Nt_4\bar{t}_2t_1, Nt_1\bar{t}_2t_0, Nt_3\bar{t}_2t_0, Nt_4\bar{t}_2t_0, N\bar{t}_0t_1\bar{t}_2, N\bar{t}_4t_1\bar{t}_2, N\bar{t}_3t_1\bar{t}_2, N\bar{t}_0t_1\bar{t}_4, N\bar{t}_3t_1\bar{t}_4, N\bar{t}_2t_1\bar{t}_4,$
$N\bar{t}_2t_1\bar{t}_0, N\bar{t}_3t_1\bar{t}_0, N\bar{t}_4t_1\bar{t}_0, N\bar{t}_0t_1\bar{t}_3, N\bar{t}_2t_1\bar{t}_3, N\bar{t}_4t_1\bar{t}_3, Nt_4\bar{t}_0t_1, Nt_2\bar{t}_0t_1, Nt_3\bar{t}_0t_1, Nt_4\bar{t}_0t_2,$
$Nt_3\bar{t}_0t_2, Nt_1\bar{t}_0t_2, Nt_1\bar{t}_0t_4, Nt_3\bar{t}_0t_4, Nt_2\bar{t}_0t_4, Nt_4\bar{t}_0t_3, Nt_1\bar{t}_0t_3, Nt_2\bar{t}_0t_3, Nt_2\bar{t}_3t_4, Nt_0\bar{t}_3t_4,$
$Nt_1\bar{t}_3t_4, Nt_2\bar{t}_3t_0, Nt_1\bar{t}_3t_0, Nt_4\bar{t}_3t_0, Nt_4\bar{t}_3t_2, Nt_1\bar{t}_3t_2, Nt_0\bar{t}_3t_2, Nt_2\bar{t}_3t_1, Nt_4\bar{t}_3t_1, Nt_0\bar{t}_3t_1,$
$N\bar{t}_1t_2\bar{t}_4, N\bar{t}_3t_2\bar{t}_4, N\bar{t}_0t_2\bar{t}_4, N\bar{t}_1t_2\bar{t}_3, N\bar{t}_0t_2\bar{t}_3, N\bar{t}_4t_2\bar{t}_3, N\bar{t}_4t_2\bar{t}_1, N\bar{t}_0t_2\bar{t}_1, N\bar{t}_3t_2\bar{t}_1, N\bar{t}_1t_2\bar{t}_0,$
$N\bar{t}_4t_2\bar{t}_0, N\bar{t}_3t_2\bar{t}_0, N\bar{t}_4t_0\bar{t}_1, N\bar{t}_3t_0\bar{t}_1, N\bar{t}_2t_0\bar{t}_1, N\bar{t}_4t_0\bar{t}_3, N\bar{t}_2t_0\bar{t}_3, N\bar{t}_1t_0\bar{t}_3, N\bar{t}_1t_0\bar{t}_4, N\bar{t}_2t_0\bar{t}_4,$
$N\bar{t}_3t_0\bar{t}_4, N\bar{t}_4t_0\bar{t}_2, N\bar{t}_1t_0\bar{t}_2, N\bar{t}_3t_0\bar{t}_2, Nt_3\bar{t}_4t_0, Nt_1\bar{t}_4t_0, Nt_2\bar{t}_4t_0, Nt_3\bar{t}_4t_1, Nt_2\bar{t}_4t_1, Nt_0\bar{t}_4t_1,$
$Nt_0\bar{t}_4t_3, Nt_2\bar{t}_4t_3, Nt_1\bar{t}_4t_3, Nt_3\bar{t}_4t_2, Nt_0\bar{t}_4t_2, Nt_1\bar{t}_4t_2, N\bar{t}_2t_4\bar{t}_3, N\bar{t}_0t_4\bar{t}_3, N\bar{t}_1t_4\bar{t}_3, N\bar{t}_2t_4\bar{t}_0,$
$N\bar{t}_1t_4\bar{t}_0, N\bar{t}_3t_4\bar{t}_0, N\bar{t}_3t_4\bar{t}_2, N\bar{t}_1t_4\bar{t}_2, N\bar{t}_0t_4\bar{t}_2, N\bar{t}_2t_4\bar{t}_1, N\bar{t}_3t_4\bar{t}_1, N\bar{t}_0t_4\bar{t}_1, N\bar{t}_2t_3\bar{t}_0, N\bar{t}_4t_3\bar{t}_0,$
$N\bar{t}_1t_3\bar{t}_0, N\bar{t}_2t_3\bar{t}_4, N\bar{t}_1t_3\bar{t}_4, N\bar{t}_0t_3\bar{t}_4, N\bar{t}_0t_3\bar{t}_2, N\bar{t}_1t_3\bar{t}_2, N\bar{t}_4t_3\bar{t}_2, N\bar{t}_2t_3\bar{t}_1, N\bar{t}_0t_3\bar{t}_1, N\bar{t}_4t_3\bar{t}_1\}$

Next, we must take a representative from each orbit of $N^{(0\bar{1}2)}$ and apply it to $Nt_0\bar{t}_1t_2$ to determine if it will extend or collapse the Cayley graph.

$Nt_0\bar{t}_1t_2 \cdot t_0 = Nt_0\bar{t}_1t_2t_0 = Nt_0\bar{t}_1t_2 \in [0\bar{1}2]$.
Since the orbit of 0 is of length 1,
one t_i will map $[0\bar{1}2]$ to itself

$Nt_0\bar{t}_1t_2 \cdot t_1 = Nt_0\bar{t}_1t_2t_1 = Nt_0t_1t_2 \in [012]$.
Since the orbit of 1 is of length 1,
one t_i will map from $[0\bar{1}2]$ to $[012]$.

$Nt_0\bar{t}_1t_2 \cdot t_2 = Nt_0\bar{t}_1t_2^2 = Nt_0\bar{t}_1\bar{t}_2 = Nt_0t_1t_2 \in [012]$.
Since the orbit of 2 is of length 1,
one t_i will map $[0\bar{1}2]$ to$[012]$

$Nt_0\bar{t}_1t_2 \cdot t_3 = Nt_0\bar{t}_1t_2t_3 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of 3 is of length 1,
one t_i will map from $[0\bar{1}2]$ to $[012\bar{3}]$.

$Nt_0\bar{t}_1t_2 \cdot t_4 = Nt_0\bar{t}_1t_2t_4 = Nt_0t_1t_2\bar{t}_4 \in [012\bar{4}]$.
Since the orbit of 4 is of length 1,
one t_i will map from $[0\bar{1}2]$ to $[012\bar{4}]$.

$Nt_0\bar{t}_1t_2 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_0 \in [0\bar{1}2\bar{0}]$.
Since the orbit of $\bar{0}$ is of length 1,
one t_i will map $[0\bar{1}2]$ to $[0\bar{1}2\bar{0}]$

$Nt_0\bar{t}_1t_2 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_2\bar{t}_1 = Nt_0t_1t_2\bar{t}_0 \in [012\bar{0}]$.
Since the orbit of $\bar{1}$ is of length 1,
one t_i will map from $[0\bar{1}2]$ to $[012\bar{0}]$.

$Nt_0\bar{t}_1t_2 \cdot \bar{t}_2 = Nt_0\bar{t}_1(t_2\bar{t}_2) = Nt_0\bar{t}_1 \in [0\bar{1}]$.
Since the orbit of $\bar{2}$ is of length 1,
one t_i will map $[0\bar{1}2]$ to$[0\bar{1}]$

$Nt_0\bar{t}_1t_2 \cdot \bar{t}_3 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of $\bar{3}$ is of length 1,

one t_i will map from $[0\bar{1}2]$ to $[0\bar{1}2\bar{3}]$.

$Nt_0\bar{t}_1t_2 \cdot \bar{t}_4 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of $\bar{4}$ is of length 1,
one t_i will map from $[0\bar{1}2]$ to $[0\bar{1}2\bar{4}]$.

Now we look at the double coset $Nt_0t_1t_2t_3N$ which we denoted by [0123]. To obtain the elements in [0123] we must first find the point stabiliser of $0, 1, 2$, and 3, N^{0123}. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$ and permute 4, hence $\left|N^{0123}\right| = 1$. Since there are no additional relations which would increase these permutations, we get that the set stabiliser $N^{(0123)} = N^{0123}$. Therefore, $\left|N^{(0123)}\right| = 1$. To find the orbits of $N^{(0123)}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(0123)}$. The orbits of $N^{(0123)}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset [0123] is obtained by dividing $\frac{|N|}{\left|N^{(0123)}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5, as done in the previous double cosets:
$[0123] = Nt_0t_1t_2t_3N = \{N(t_0t_1t_2t_3)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(0123)}$ and apply it to $Nt_0t_1t_2t_3$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3 \cdot t_0 = Nt_0t_1t_2t_3t_0$.
This is a new double coset which we will label as [01230].
Since the orbit of 0 is of length 1,
one t_i will map from [0123] to [01230].

$Nt_0t_1t_2t_3 \cdot t_1 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of 1 is of length 1,
one t_i will map from [0123] to $[0\bar{1}2\bar{4}]$.

$Nt_0t_1t_2t_3 \cdot t_2 = Nt_0t_1t_2t_3t_2 = Nt_0t_1t_2 \in [012]$.
Since the orbit of 2 is of length 1,

one t_i will map from [0123] to [012].

$Nt_0t_1t_2t_3 \cdot t_3 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of 3 is of length 1,
one t_i will map from [0123] to $[012\bar{3}]$.

$Nt_0t_1t_2t_3 \cdot t_4 = Nt_0t_1t_2t_3t_4$.
This is a new double coset which we will label as [01234].
Since the orbit of 4 is of length 1,
one t_i will extend the Cayley graph from [0123] to [01234].

$Nt_0t_1t_2t_3 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will map the Cayley graph from [0123] to $[0\bar{1}2\bar{3}]$.

$Nt_0t_1t_2t_3 \cdot \bar{t}_1 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the Cayley graph from [0123] to [01230].

Notice that there are now two things which expand the Cayley graph from [0123] to [01230]

$Nt_0t_1t_2t_3 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_4 \in [012\bar{4}]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will collapse the Cayley graph from [012] to [01].

$Nt_0t_1t_2t_3 \cdot \bar{t}_3 = Nt_0t_1t_2 \in [012]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will collapse the Cayley graph from [0123] to [012].

$Nt_0t_1t_2t_3 \cdot \bar{t}_4 = Nt_0t_1t_2t_3\bar{t}_4$.
This is a new double coset which we will label as $[0123\bar{4}]$.

Since the orbit of $\bar{4}$ is of length one,
one t_i will expand the Cayley graph from [0123] to [0123$\bar{4}$].

Now we look at the double coset $Nt_0t_1t_2\bar{t}_3$ which we denoted by [012$\bar{3}$]. To obtain the elements in [012$\bar{3}$], we must first find the point stabiliser of $0, 1, 2$, and $\bar{3}$, $N^{012\bar{3}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, \bar{3}$ and permute 4, hence $\left|N^{012\bar{3}}\right| = 1$. Since there are no additional relations which would increase these permutations, we get that the set stabiliser $N^{(012\bar{3})} = N^{012\bar{3}}$. Therefore, $\left|N^{(012\bar{3})}\right| = 1$. To find the orbits of $N^{(012\bar{3})}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(012\bar{3})}$. The orbits of $N^{(012\bar{3})}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset [012$\bar{3}$] is obtained by dividing $\frac{|N|}{\left|N^{(012\bar{3})}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[012\bar{3}] = Nt_0t_1t_2\bar{t}_3N = \{N\,(t_0t_1t_2\bar{t}_3)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(012\bar{3})}$ and apply it to $Nt_0t_1t_2\bar{t}_3$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_3 \cdot t_0 = Nt_0t_1t_2\bar{t}_3t_0$.
This is a new double coset which we will label as [012$\bar{3}$0].
Since the orbit of 0 is of length 1,
one t_i will map from [012$\bar{3}$] to [012$\bar{3}$0].

$Nt_0t_1t_2\bar{t}_3 \cdot t_1 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of 1 is of length 1,
one t_i will map from [012$\bar{3}$] to [0$\bar{1}$2$\bar{3}$].

$Nt_0t_1t_2\bar{t}_3 \cdot t_2 = Nt_0t_1t_2\bar{t}_3t_2$.
This is a new double coset which we will label as [012$\bar{3}$2].
Since the orbit of 2 is of length 1,
one t_i will extend the Cayley graph from [012$\bar{3}$] to [012$\bar{3}$2].

$Nt_0t_1t_2\bar{t}_3 \cdot t_3 = Nt_0t_1t_2 \in [012]$.

Since the orbit of 3 is of length 1,
one t_i will map from $[012\bar{3}]$ to $[012]$.

$Nt_0t_1t_2\bar{t}_3 \cdot t_4 = Nt_0t_1t_2\bar{t}_3t_4$.
This is a new double coset which we will label as $[012\bar{3}4]$.
Since the orbit of 4 is of length 1,
one t_i will extend the Cayley graph from $[012\bar{3}]$ to $[012\bar{3}4]$.

$Nt_0t_1t_2\bar{t}_3 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{3}]$ to $[0\bar{1}2\bar{3}]$.
Notice that there are now two t_i's which map the Cayley graph
from $[012\bar{3}]$ to $[0\bar{1}2\bar{3}]$.

$Nt_0t_1t_2\bar{t}_3 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_2 \cdot \in [0\bar{1}2]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{3}]$ to $[0\bar{1}2]$.

$Nt_0t_1t_2\bar{t}_3 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{3}]$ to $[012\bar{3}]$

$Nt_0t_1t_2\bar{t}_3 \cdot \bar{t}_3 = Nt_0t_1t_2t_3 \in [0123]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{3}]$ to $[0123]$

$Nt_0t_1t_2\bar{t}_3 \cdot \bar{t}_4 = Nt_0t_1t_2t_3\bar{t}_4 \in [0123\bar{4}]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{3}]$ to $[0123\bar{4}]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_4N$ which we denoted by $[012\bar{4}]$. To obtain the elements in $[012\bar{4}]$, we must first find the point stabiliser of $0, 1, 2$, and $\bar{4}$,

$N^{012\bar{4}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, \bar{4}$ and permute 3, hence $\left|N^{012\bar{4}}\right| = 1$. Since there are no additional relations which would increase these permutations, we get that the set stabiliser $N^{(012\bar{4})} = N^{012\bar{4}}$. Therefore, $\left|N^{(012\bar{4})}\right| = 1$. To find the orbits of $N^{(012\bar{4})}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(012\bar{4})}$. The orbits of $N^{(012\bar{4})}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[012\bar{4}]$ is obtained by dividing $\frac{|N|}{\left|N^{(012\bar{4})}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[012\bar{4}] = Nt_0t_1t_2\bar{t}_4N = \{N(t_0t_1t_2\bar{t}_4)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(012\bar{4})}$ and apply it to $Nt_0t_1t_2\bar{t}_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_4 \cdot t_0 = Nt_0t_1t_2\bar{t}_4t_0$.
This is a new double coset which we will label as $[012\bar{4}0]$.
Since the orbit of 0 is of length 1,
one t_i will expand the cayley graph from $[012\bar{4}]$ to $[012\bar{4}0]$.

$Nt_0t_1t_2\bar{t}_4 \cdot t_1 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of 1 is of length 1,
one t_i will map from $[012\bar{4}]$ to $[0\bar{1}2\bar{4}]$.

$Nt_0t_1t_2\bar{t}_4 \cdot t_2 = Nt_0t_1t_2\bar{t}_3t_2 \in [012\bar{3}2]$.
Since the orbit of 2 is of length 1,
one t_i will map from $[012\bar{4}]$ to $[012\bar{3}2]$.

$Nt_0t_1t_2\bar{t}_4 \cdot t_3 = Nt_0t_1t_2\bar{t}_4t_3$.
This is a new double coset which we will label as $[012\bar{4}3]$
Since the orbit of 3 is of length 1,
one t_i will expand the cayley graph from $[012\bar{4}]$ to $[012\bar{4}3]$.

$Nt_0t_1t_2\bar{t}_4 \cdot t_4 = Nt_0t_1t_2 \in [012]$.
Since the orbit of 4 is of length 1,
one t_i will collapse the Cayley graph from $[012\bar{4}]$ to $[012]$.

$Nt_0t_1t_2\bar{t}_4 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will map from $[012\bar{4}]$ to $[0\bar{1}2\bar{4}]$.
Notice that there are now two t_i's which will map
from $[012\bar{4}]$ to $[0\bar{1}2\bar{4}]$.

$Nt_0t_1t_2\bar{t}_4 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_2 \in [0\bar{1}2]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will collapse the Cayley graph from $[012\bar{4}]$ to $[0\bar{1}2]$.

$Nt_0t_1t_2\bar{t}_4 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_4 \in [012\bar{4}]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will map from $[012\bar{4}]$ to $[012\bar{4}]$

$Nt_0t_1t_2\bar{t}_4 \cdot \bar{t}_3 = Nt_0t_1t_2t_3t_4 \in [01234]$.

Since the orbit of $\bar{3}$ is of length one,
one t_i will expand the Cayley graph from $[012\bar{4}]$ to $[01234]$.

$Nt_0t_1t_2\bar{t}_4 \cdot \bar{t}_4 = Nt_0t_1t_2t_4 = Nt_0t_1t_2t_3 \in [0123]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will map from $[012\bar{4}]$ to $[0123]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_0N$ which we denoted by $[012\bar{0}]$. To obtain the elements in $[012\bar{0}]$, we must first find the point stabiliser of $0, 1, 2$, and $\bar{0}$, $N^{012\bar{0}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, \bar{0}$ and permute 3.4, hence $\left|N^{012\bar{0}}\right| = 2$. Since there are no additional relations which would increase these permutations, we get that the set stabiliser $N^{(012\bar{0})} = N^{012\bar{0}}$. Therefore, $\left|N^{(012\bar{0})}\right| = 2$. To find the orbits of $N^{(012\bar{0})}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(012\bar{0})}$. The orbits of $N^{(012\bar{0})}$ are $\{0, 2\}, \{1\}, \{3, 4\}, \{\bar{0}, \bar{2}\}, \{\bar{1}\}$, and $\{\bar{3}, \bar{4}\}$. The number of single cosets in the double coset $[012\bar{0}]$ is obtained by dividing $\frac{|N|}{\left|N^{(012\bar{0})}\right|} = \frac{120}{2} = 60$. To obtain

the single cosets in this double coset we apply the permutations in S_5:
$[012\bar{0}] = Nt_0t_1t_2\bar{t}_0N = \{N\,(t_0t_1t_2\bar{t}_0)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(012\bar{0})}$ and apply it to $Nt_0t_1t_2\bar{t}_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_0 \cdot t_0 = Nt_0t_1t_2 \in [012]$.
Since the orbit of 0 is of length two,
two t_i's will collapse the cayley graph from $[012\bar{0}]$ to $[012]$.

$Nt_0t_1t_2\bar{t}_0 \cdot t_1 = Nt_0t_1t_2\bar{t}_0 \in [012\bar{0}]$.
Since the orbit of 1 is of length one,
one t_i will map from $[012\bar{0}]$ to $[012\bar{0}]$.

$Nt_0t_1t_2\bar{t}_0 \cdot t_3 = Nt_0t_1t_2\bar{t}_3t_2 \in [012\bar{3}2]$.
Since the orbit of 3 is of length two,
two t_i 's will expand the cayley graph from $[012\bar{0}]$ to $[012\bar{3}2]$.

$Nt_0t_1t_2\bar{t}_0 \cdot \bar{t}_0 = Nt_0t_1t_2t_0 = Nt_0\bar{t}_1t_2 \in [0\bar{1}2]$.
Since the orbit of $\bar{0}$ is of length two,
two t_i 's will collapse the cayley graph from $[012\bar{0}]$ to $[0\bar{1}2]$.

$Nt_0t_1t_2\bar{t}_0 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_0 \in [0\bar{1}0]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will collapse the Cayley graph from $[012\bar{0}]$ to $[0\bar{1}0]$.

$Nt_0t_1t_2\bar{t}_0 \cdot \bar{t}_3 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of $\bar{3}$ is of length two,
two t_i 's will expand the Cayley graph from $[012\bar{0}]$ to $[01230]$.

Now we look at the double coset $Nt_0\bar{t}_1t_2\bar{t}_3N$ which we denoted by $[0\bar{1}2\bar{3}]$. To obtain the elements in $[0\bar{1}2\bar{3}]$, we must first find the point stabiliser of $0, \bar{1}, 2$, and $\bar{3}$, $N^{0\bar{1}2\bar{3}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, \bar{1}, 2, \bar{3}$ and

permute 4, hence $\left|N^{0\bar{1}2\bar{3}}\right| = 1$. Since there are no additional relations which would increase these permutations, we get that the set stabiliser $N^{(0\bar{1}2\bar{3})} = N^{0\bar{1}2\bar{3}}$. Therefore, $\left|N^{(0\bar{1}2\bar{3})}\right| = 1$. To find the orbits of $N^{(0\bar{1}2\bar{3})}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(0\bar{1}2\bar{3})}$. The orbits of $N^{(0\bar{1}2\bar{3})}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$ and $\{\bar{4}\}$. The number of single cosets in the double coset $[0\bar{1}2\bar{3}]$ is obtained by dividing $\frac{|N|}{|N^{(0\bar{1}2\bar{3})}|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[0\bar{1}2\bar{3}] = Nt_0\bar{t}_1t_2\bar{t}_3N = \{N\,(t_0\bar{t}_1t_2\bar{t}_3)^n\,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(0\bar{1}2\bar{3})}$ and apply it to $Nt_0\bar{t}_1t_2\bar{t}_3$ to determine if it will extend or collapse the Cayley graph.

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_0$.
This is a new double coset which we will label as $[0\bar{1}2\bar{3}0]$.
Since the orbit of 0 is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{3}]$ to $[0\bar{1}2\bar{3}0]$.

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot t_1 = Nt_0t_1t_2\bar{t}_3t_0 \in [012\bar{3}0]$.
Since the orbit of 1 is of length one,
one t_i will map from $[0\bar{1}2\bar{3}]$ to $[012\bar{3}0]$.

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot t_2 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of 2 is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{3}]$ to $[01230]$

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot t_3 = Nt_0\bar{t}_1t_2 \in [0\bar{1}2]$.
Since the orbit of 3 is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{3}]$ to $[0\bar{1}2]$.

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot t_4 = Nt_0\bar{t}_1t_2\bar{t}_3t_4$.
This is a new double coset which we will label as $[0\bar{1}2\bar{3}4]$.
Since the orbit of 4 is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{3}]$ to $[0\bar{1}2\bar{3}4]$

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{3}]$ to $[0\bar{1}2\bar{4}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot \bar{t}_1 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{3}]$ to $[012\bar{3}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot \bar{t}_2 = Nt_0t_1t_2t_3 \in [0123]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{3}]$ to $[0123]$

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{3}]$ to $[012\bar{3}]$.
Now, two things will map from $[0\bar{1}2\bar{3}]$ to $[012\bar{3}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3 \cdot \bar{t}_4 = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{3}]$ to $[012\bar{4}3]$

Now we look at the double coset $Nt_0\bar{t}_1t_2\bar{t}_4N$ which we denoted by $[0\bar{1}2\bar{4}]$. To obtain the elements in $[0\bar{1}2\bar{4}]$, we must first find the point stabiliser of $0, \bar{1}, 2$, and $\bar{4}$, $N^{0\bar{1}2\bar{4}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, \bar{1}, 2, \bar{4}$ and permute 3, hence $\left|N^{0\bar{1}2\bar{4}}\right| = 1$. Since there are no additional relations which would increase these permutations, we get that the set stabiliser $N^{(0\bar{1}2\bar{4})} = N^{0\bar{1}2\bar{4}}$. Therefore, $\left|N^{(0\bar{1}2\bar{4})}\right| = 1$. To find the orbits of $N^{(0\bar{1}2\bar{4})}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(0\bar{1}2\bar{4})}$. The orbits of $N^{(0\bar{1}2\bar{4})}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$ and $\{\bar{4}\}$. The number of single cosets in the double coset $[0\bar{1}2\bar{4}]$ is obtained by dividing $\frac{|N|}{\left|N^{(0\bar{1}2\bar{4})}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[0\bar{1}2\bar{4}] = Nt_0\bar{t}_1t_2\bar{t}_4N = \{N\,(t_0\bar{t}_1t_2\bar{t}_4)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(0\bar{1}2\bar{4})}$ and apply it to

$Nt_0\bar{t}_1t_2\bar{t}_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_0 \in [0\bar{1}2\bar{3}0]$.
Since the orbit of 0 is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{4}]$ to $[0\bar{1}2\bar{3}0]$.

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot t_1 = Nt_0t_1t_2\bar{t}_4t_0 \in [012\bar{4}0]$.
Since the orbit of 1 is of length one,
one t_i will map from $[0\bar{1}2\bar{4}]$ to $[012\bar{4}0]$.

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot t_2 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of 2 is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{4}]$ to $[01230]$

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot t_3 = Nt_0\bar{t}_1t_2\bar{t}_4t_3$
This is a new double coset which we will label as $[0\bar{1}2\bar{4}3]$
Since the orbit of 3 is of length one,
one t_i will expand the cayley graph from $[0\bar{1}24]$ to $[0\bar{1}2\bar{4}3]$.

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot t_4 = Nt_0\bar{t}_1t_2 \in [0\bar{1}2]$.
Since the orbit of 4 is of length one,
one t_i will collapse the cayley graph from $[0\bar{1}2\bar{4}]$ to $[0\bar{1}2]$

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{4}]$ to $[0\bar{1}2\bar{3}]$.

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot \bar{t}_1 = Nt_0t_1t_2\bar{t}_4 \in [012\bar{4}]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{4}]$ to $[012\bar{4}]$.

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot \bar{t}_2 = Nt_0t_1t_2t_3 \in [0123]$.

Since the orbit of $\bar{2}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{4}]$ to $[0123]$

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{4}]$ to $[012\bar{3}4]$.

$Nt_0\bar{t}_1t_2\bar{t}_4 \cdot \bar{t}_4 = Nt_0t_1t_2\bar{t}_4 \in [012\bar{4}]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will map from $[0\bar{1}2\bar{4}]$ to $[012\bar{4}]$
Now, two things will map from $[0\bar{1}2\bar{4}]$ to $[012\bar{4}]$.

Now we look at the double coset $Nt_0\bar{t}_1t_2\bar{t}_0N$ which we denoted by $[0\bar{1}2\bar{0}]$. To obtain the elements in $[0\bar{1}2\bar{0}]$, we must first find the point stabiliser of $0, \bar{1}, 2$, and $\bar{0}$, $N^{0\bar{1}2\bar{0}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, \bar{1}, 2, \bar{0}$ and permute $3, 4$, hence $\left|N^{0\bar{1}2\bar{0}}\right| = 2$.
Take the relation $t_0t_1 = t_1t_0$ and conjugate it by $(1\bar{1}) \in N$ to get
$(t_0t_1)^{(1\bar{1})} = (t_1t_0)^{(1\bar{1})} \Rightarrow t_0\bar{t}_1 = \bar{t}_1t_0$
Now, multiply on the right by t_2:
$\Rightarrow t_0\bar{t}_1t_2 = \bar{t}_1t_0t_2$
Multiply on the right by $\bar{t}_0$
$\Rightarrow t_0\bar{t}_1t_2\bar{t}_0 = \bar{t}_1t_0t_2\bar{t}_0$
$\Rightarrow Nt_0\bar{t}_1t_2\bar{t}_0 = N\bar{t}_1t_0t_2\bar{t}_0$.
Now, take the relation $t_0t_1 = t_1t_0$ and conjugate it by $(0\bar{1})\,(12) \in N$.
$(t_0t_1)^{(0\bar{1})(12)} = (t_1t_0)^{(0\bar{1})(12)} \Rightarrow \bar{t}_1t_2 = t_2\bar{t}_1$.
Multiply on the left by t_0
$\Rightarrow t_0\bar{t}_1t_2 = t_0t_2\bar{t}_1$.
Multiply on the right by $\bar{t}_0$
$\Rightarrow t_0\bar{t}_1t_2\bar{t}_0 = t_0t_2\bar{t}_1\bar{t}_0$
$\Rightarrow Nt_0\bar{t}_1t_2\bar{t}_0 = Nt_0t_2\bar{t}_1\bar{t}_0$.
Let $(02\bar{1})$ be a permutation in N.
$Nt_0\bar{t}_1t_2\bar{t}_0{}^{(02\bar{1})} = Nt_2t_0\bar{t}_1\bar{t}_0$. If we conjugate the relation $t_0t_1 = t_1t_0$ by $(12) \in N$, we get

$t_0 t_2 = t_2 t_0$.

$\Longrightarrow Nt_2 t_0 \bar{t}_1 \bar{t}_0 = Nt_0 t_2 \bar{t}_1 \bar{t}_0$ but $Nt_0 \bar{t}_1 t_2 \bar{t}_0 = Nt_0 t_2 \bar{t}_1 \bar{t}_0$

$\Longrightarrow Nt_2 t_0 \bar{t}_1 \bar{t}_0 = Nt_0 \bar{t}_1 t_2 \bar{t}_0$

Now, $(02\bar{1}) \in N^{(0\bar{1}2\bar{0})}$.

So, $N^{(0\bar{1}2\bar{0})} \geq \left\langle (02\bar{1}), N^{0\bar{1}2\bar{0}} \right\rangle \Rightarrow \left| N^{(0\bar{1}2\bar{0})} \right| = 6$.

To find the orbits of $N^{(0\bar{1}2\bar{0})}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(0\bar{1}2\bar{0})}$. The orbits of $N^{(0\bar{1}2\bar{0})}$ are $\{0, 2, 1, \bar{0}, \bar{1}, \bar{2}\}$, $\{3, \bar{4}\}$, and $\{4, \bar{3}\}$.The number of single cosets in the double coset $[0\bar{1}2\bar{0}]$ is obtained by dividing $\frac{|N|}{\left|N^{(0\bar{1}2\bar{0})}\right|} = \frac{120}{6} = 20$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[0\bar{1}2\bar{0}] = Nt_0 \bar{t}_1 t_2 \bar{t}_0 N = \{N (t_0 \bar{t}_1 t_2 \bar{t}_0)^n \, | n \in N\}$

Next, we must take a representative of each orbit of $N^{(0\bar{1}2\bar{0})}$ and apply it to $Nt_0 \bar{t}_1 t_2 \bar{t}_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0 \bar{t}_1 t_2 \bar{t}_0 \cdot t_0 = Nt_0 \bar{t}_1 t_2 \in [0\bar{1}2]$.
Since the orbit of 0 is of length six,
six t_i's will collapse the cayley graph from $[0\bar{1}2\bar{0}]$ to $[0\bar{1}2]$.

$Nt_0 \bar{t}_1 t_2 \bar{t}_0 \cdot t_3 = Nt_0 t_1 t_2 \bar{t}_4 t_0 \in [012\bar{4}0]$
Since the orbit of 3 is of length two,
two t_i 's will expand the cayley graph from $[0\bar{1}2\bar{0}]$ to $[012\bar{4}0]$.

$Nt_0 \bar{t}_1 t_2 \bar{t}_0 \cdot t_4 = Nt_0 t_1 t_2 \bar{t}_3 t_0 \in [012\bar{3}0]$.
Since the orbit of 4 is of length two,
two t_i 's will expand the cayley graph from $[0\bar{1}2\bar{0}]$ to $[012\bar{3}0]$

Now we look at the double coset $Nt_0 t_1 t_2 t_3 t_0 N$ which we denoted by [01230]. To obtain the elements in [01230], we must first find the point stabiliser of $0, 1, 2$, and 3, N^{01230}. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$ and permute 4, hence $\left|N^{01230}\right| = 1$. Since there are no additional relations which would increase these permutations, we obtain that the set stabiliser $N^{(01230)} = N^{01230}$, and so $\left|N^{(01230)}\right| = 1$.. To find the orbits of $N^{(01230)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(01230)}$. The orbits of $N^{(01230)}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$,

and $\{\bar{4}\}$.The number of single cosets in the double coset [01230] is obtained by dividing $\frac{|N|}{|N^{(01230)}|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[01230] = Nt_0t_1t_2t_3t_0N = \{N\,(t_0t_1t_2t_3t_0)^n\,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(01230)}$ and apply it to $Nt_0t_1t_2t_3t_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3t_0 \cdot t_0 = Nt_0t_1t_2t_3\bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of 0 is of length one,
one t_i will collapse the cayley graph from [01230] to $[0\bar{1}23]$.

$Nt_0t_1t_2t_3t_0 \cdot t_1 = Nt_0t_1t_2t_3t_0t_1$.
This is a new double coset which we will label as [012301].
Since the orbit of 1 is of length one,
one t_i will expand the cayley graph from [01230] to [012301].

$Nt_0t_1t_2t_3t_0 \cdot t_2 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of 2 is of length one,
one t_i will collapse the cayley graph from [01230] to $[0\bar{1}2\bar{4}]$.

$Nt_0t_1t_2t_3t_0 \cdot t_3 = Nt_0t_1t_2\bar{t}_0 \in [0\bar{1}2\bar{0}]$.
Since the orbit of 3 is of length one,
one t_i will collapse the cayley graph from [01230] to $[0\bar{1}2\bar{0}]$.

$Nt_0t_1t_2t_3t_0 \cdot t_4 = Nt_0t_1t_2t_3t_0t_4$.
This is a new double coset which we will label as [012304]
Since the orbit of 4 is of length one,
one t_i will expand the cayley graph from [01230] to [012304].

$Nt_0t_1t_2t_3t_0 \cdot \bar{t}_0 = Nt_0t_1t_2t_3 \in [0123]$.
Since the orbit of $\bar{0}$ is of length one,

one t_i will collapse the cayley graph from [01230] to [0123]

$Nt_0t_1t_2t_3t_0 \cdot \bar{t}_1 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the cayley graph from [01230] to [01230].

$Nt_0t_1t_2t_3t_0 \cdot \bar{t}_2 = Nt_0t_1t_2t_3 \in [0123]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will collapse the cayley graph from [01230] to [0123].
Now there are two things that collapse the cayley graph
from [01230] to [0123].

$Nt_0t_1t_2t_3t_0 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_3t_2 \in [012\bar{3}2]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will map the cayley graph from [01230] to $[012\bar{3}2]$.

$Nt_0t_1t_2t_3t_0 \cdot \bar{t}_4 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [01230\bar{4}]$.
This is a new double coset which we will label as $[01230\bar{4}]$
Since the orbit of $\bar{4}$ is of length one,
one t_i will expand the cayley graph from [01230] to $[01230\bar{4}]$.

Now we look at the double coset $Nt_0t_1t_2t_3t_4N$ which we denoted by [01234]. To obtain the elements in [01234], we must first find the point stabiliser of $0, 1, 2, 3$ and 4, N^{01234}. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3, 4$, hence $\left|N^{01234}\right| = 1$. Since there are no additional relations which would increase these permutations, we obtain that the set stabiliser $N^{(01234)} = N^{01234}$, and so $\left|N^{(01234)}\right| = 1$. To find the orbits of $N^{(01234)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(01234)}$. The orbits of $N^{(01234)}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset [01234] is obtained by dividing $\frac{|N|}{\left|N^{(01234)}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5: $[01234] = Nt_0t_1t_2t_3t_4N = \{N(t_0t_1t_2t_3t_4)^n \,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(01234)}$ and apply it to

$Nt_0t_1t_2t_3t_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3t_4 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4 \in [0\bar{1}2\bar{3}4]$.
Since the orbit of 0 is of length one,
one t_i will collapse the cayley graph from [01234] to $[0\bar{1}2\bar{3}4]$.

$Nt_0t_1t_2t_3t_4 \cdot t_1 = Nt_0t_1t_2t_3t_0t_4 \in [012304]$
Since the orbit of 1 is of length one,
one t_i will expand the cayley graph from [01234] to [012304].

$Nt_0t_1t_2t_3t_4 \cdot t_2 = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of 2 is of length one,
one t_i will collapse the cayley graph from [01234] to $[012\bar{4}3]$.

$Nt_0t_1t_2t_3t_4 \cdot t_3 = Nt_0t_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of 3 is of length one,
one t_i will collapse the cayley graph from [01234] to $[0\bar{1}2\bar{4}]$.

$Nt_0t_1t_2t_3t_4 \cdot t_4 = Nt_0t_1t_2t_3\bar{t}_4$.
Since the orbit of 4 is of length one,
one t_i will map the cayley graph from [01234] to $[0123\bar{4}]$.

$Nt_0t_1t_2t_3t_4 \cdot \bar{t}_0 = Nt_0t_1t_2t_3t_4\bar{t}_0$.
This is a new double coset which we will label as $[01234\bar{0}]$
Since the orbit of $\bar{0}$ is of length one,
one t_i will expand the cayley graph from [01234] to $[01234\bar{0}]$

$Nt_0t_1t_2t_3t_4 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the cayley graph from [01234] to $[0\bar{1}2\bar{4}3]$.

$Nt_0t_1t_2t_3t_4 \cdot \bar{t}_2 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [01230\bar{4}]$.

Since the orbit of $\bar{2}$ is of length one,
one t_i will collapse the cayley graph from [01234] to [0123$\bar{0}$4].

$Nt_0t_1t_2t_3t_4 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will map the cayley graph from [01234] to [012$\bar{4}$3].
Now there are two things that map the cayley graph
from [01234] to [012$\bar{4}$3].

$Nt_0t_1t_2t_3t_4 \cdot \bar{t}_4 = Nt_0t_1t_2t_3 \in [0123]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will collapse the cayley graph from [01234] to [0123]

Now we look at the double coset $Nt_0t_1t_2t_3\bar{t}_4N$ which we denoted by [0123$\bar{4}$]. To obtain the elements in [0123$\bar{4}$], we must first find the point stabiliser of $0, 1, 2, 3$ and $\bar{4}$, $N^{0123\bar{4}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$, and permute $\bar{4}$ hence $\left|N^{0123\bar{4}}\right| = 1$. Since there are no additional relations which would increase these permutations, we obtain that the set stabiliser $N^{(0123\bar{4})} = N^{0123\bar{4}}$, and so $\left|N^{(0123\bar{4})}\right| = 1$. To find the orbits of $N^{(0123\bar{4})}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(0123\bar{4})}$. The orbits of $N^{(0123\bar{4})}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset [0123$\bar{4}$] is obtained by dividing $\frac{|N|}{\left|N^{(01234)}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[0123\bar{4}] = Nt_0t_1t_2t_3\bar{t}_4N = \{N\,(t_0t_1t_2t_3\bar{t}_4)^n \,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(0123\bar{4})}$ and apply it to $Nt_0t_1t_2t_3\bar{t}_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3\bar{t}_4 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4 \in [0\bar{1}2\bar{3}4]$.
Since the orbit of 0 is of length one,
one t_i will collapse the cayley graph from [0123$\bar{4}$] to [0$\bar{1}$2$\bar{3}$4].

$Nt_0t_1t_2t_3\bar{t}_4 \cdot t_1 = Nt_0t_1t_2t_3\bar{t}_4t_1$

This is a new double coset which we will label as $[012\bar{3}\bar{4}1]$
Since the orbit of 1 is of length one,
one t_i will expand the cayley graph from $[012\bar{3}\bar{4}]$ to $[012\bar{3}\bar{4}1]$

$Nt_0t_1t_2t_3\bar{t}_4 \cdot t_2 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of 2 is of length one,
one t_i will map the cayley graph from $[012\bar{3}\bar{4}]$ to $[012\bar{3}4]$.

$Nt_0t_1t_2t_3\bar{t}_4 \cdot t_3 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [01230\bar{4}]$.
Since the orbit of 3 is of length one,
one t_i will expand the cayley graph from $[012\bar{3}\bar{4}]$ to $[01230\bar{4}]$.

$Nt_0t_1t_2t_3\bar{t}_4 \cdot t_4 = Nt_0t_1t_2t_3 \in [0123]$.
Since the orbit of 4 is of length one,
one t_i will collapse the cayley graph from $[012\bar{3}\bar{4}]$ to $[0123]$.

$Nt_0t_1t_2t_3\bar{t}_4 \cdot \bar{t}_0 = Nt_0t_1t_2t_3t_0t_4$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will expand the cayley graph from $[012\bar{3}\bar{4}]$ to $[012304]$

$Nt_0t_1t_2t_3\bar{t}_4 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the cayley graph from $[012\bar{3}\bar{4}]$ to $[0\bar{1}2\bar{4}3]$.

$Nt_0t_1t_2t_3\bar{t}_4 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will collapse the cayley graph from $[012\bar{3}\bar{4}]$ to $[012\bar{3}]$.
$Nt_0t_1t_2t_3\bar{t}_4 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will map the cayley graph from $[012\bar{3}\bar{4}]$ to $[012\bar{3}4]$.
Now there are two things that map the cayley graph

from $[0123\bar{4}]$ to $[012\bar{3}4]$.

$Nt_0t_1t_2t_3\bar{t}_4 \cdot \bar{t}_4 = Nt_0t_1t_2t_3t_4 \in [01234]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will collapse the cayley graph from $[0123\bar{4}]$ to $[01234]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_3t_0N$ which we denoted by $[012\bar{3}0]$. To obtain the elements in $[012\bar{3}0]$, we must first find the point stabiliser of $0, 1, 2$, and $\bar{3}$, $N^{012\bar{3}0}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$, and permutes 4 hence $\left|N^{012\bar{3}0}\right| = 1$. Using magma, we obtain the relation $012\bar{3}0 \sim 310\bar{2}3 \sim 213\bar{0}2$. Using thiese relations, we can obtain the elements in the set stabiliser. The first relation $012\bar{3}0 \sim 310\bar{2}3$ can be represented as a permutation in S_5, this permutaion is $(032)(\bar{0}\bar{3}\bar{2})$. If we conjugate $Nt_0t_1t_2\bar{t}_3t_0$ by $(032)(\bar{0}\bar{3}\bar{2})$, we obtain $N(t_0t_1t_2\bar{t}_3t_0)^{(032)(\bar{0}\bar{3}\bar{2})} = Nt_3t_1t_0\bar{t}_2t_3$, but the relation tells us that $310\bar{2}3 \sim 012\bar{3}0$
$\Rightarrow Nt_3t_1t_0\bar{t}_2t_3 = Nt_0t_1t_2\bar{t}_3t_0$
$\Rightarrow (032)(\bar{0}\bar{3}\bar{2}) \in N^{(012\bar{3}0)}$
So, $N^{(012\bar{3}0)} \geq \left\langle (032)(\bar{0}\bar{3}\bar{2}), N^{012\bar{3}0} \right\rangle \Rightarrow \left|N^{(012\bar{3}0)}\right| = 3$.
To find the orbits of $N^{(012\bar{3}0)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(012\bar{3}0)}$. The orbits of $N^{(012\bar{3}0)}$ are $\{0, 3, 2\}, \{1\}, \{4\}, \{\bar{0}, \bar{3}, \bar{2}\}, \{\bar{1}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[012\bar{3}0]$ is obtained by dividing $\frac{|N|}{\left|N^{(012\bar{3}0)}\right|} = \frac{120}{3} = 40$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[012\bar{3}0] = Nt_0t_1t_2\bar{t}_3t_0N = \{N(t_0t_1t_2\bar{t}_3t_0)^n \,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(012\bar{3}0)}$ and apply it to $Nt_0t_1t_2\bar{t}_3t_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_3t_0 \cdot t_0 = Nt_0t_1t_2\bar{t}_3\bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of 0 is of length three,
three t_i 's will collapse the cayley graph from $[012\bar{3}0]$ to $[0\bar{1}2\bar{3}]$.

$Nt_0t_1t_2\bar{t}_3t_0 \cdot t_1 = Nt_0\bar{t}_1t_2\bar{t}_0 \in [0\bar{1}2\bar{0}]$.
Since the orbit of 1 is of length one,
one t_i will collapse the cayley graph from $[012\bar{3}0]$ to $[0\bar{1}2\bar{0}]$.

$Nt_0t_1t_2\bar{t}_3t_0 \cdot t_4 = Nt_0t_1t_2t_3\bar{t}_4t_1 \in [0123\bar{4}1]$.
Since the orbit of 4 is of length one,
one t_i will expand the cayley graph from $[012\bar{3}0]$ to $[0123\bar{4}1]$.

$Nt_0t_1t_2\bar{t}_3t_0 \cdot \bar{t}_0 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of $\bar{0}$ is of length three,
three t_i will collapse the cayley graph from $[012\bar{3}0]$ to $[012\bar{3}]$.

$Nt_0t_1t_2\bar{t}_3t_0 \cdot \bar{t}_1 = Nt_0t_1t_2\bar{t}_4t_0 \in [012\bar{4}0]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the cayley graph from $[012\bar{3}0]$ to $[012\bar{4}0]$.

$Nt_0t_1t_2\bar{t}_3t_0 \cdot \bar{t}_4 = Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4$
This is a new double coset which we will label as $[012\bar{3}0\bar{4}]$
Since the orbit of $\bar{4}$ is of length one,
one t_i will expand the cayley graph from $[012\bar{3}0]$ to $[012\bar{3}0\bar{4}]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_3t_2N$ which we denoted by $[012\bar{3}2]$. To obtain the elements in $[012\bar{3}2]$, we must first find the point stabiliser of $0, 1, 2$, and $\bar{3}$, $N^{012\bar{3}2}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$, and permutes 4 hence $\left|N^{012\bar{3}2}\right| = 1$. Using magma, we obtain the relation $012\bar{3}2 \sim \bar{1}\bar{0}2\bar{3}\bar{2}$. Using these relations, we can obtain the elements in the set stabiliser. This relation can be represented as a permutation in S_5, this permutaion is $(0\bar{1})\,(1\bar{0})\,(2\bar{2})\,(3\bar{3})\,(4\bar{4})$. If we conjugate $Nt_0t_1t_2\bar{t}_3t_2$ by $(0\bar{1})\,(1\bar{0})\,(2\bar{2})\,(3\bar{3})\,(4\bar{4})$, we obtain $N\left(t_0t_1t_2\bar{t}_3t_2\right)^{(0\bar{1})(1\bar{0})(2\bar{2})(3\bar{3})(4\bar{4})} = N\bar{t}_1\bar{t}_0\bar{t}_2t_3\bar{t}_2$, but the relation tells us that $\bar{1}\bar{0}2\bar{3}\bar{2} \sim 012\bar{3}2$

$\Rightarrow N\bar{t}_1\bar{t}_0\bar{t}_2t_3\bar{t}_2 = Nt_0t_1t_2\bar{t}_3t_2$

$\Rightarrow (0\bar{1})\,(1\bar{0})\,(2\bar{2})\,(3\bar{3})\,(4\bar{4}) \in N^{(012\bar{3}2)}$

So, $N^{(012\bar{3}2)} \geq \left\langle (0\bar{1})\,(1\bar{0})\,(2\bar{2})\,(3\bar{3})\,(4\bar{4}), N^{012\bar{3}2} \right\rangle \Rightarrow \left|N^{(012\bar{3}2)}\right| = 2$. To find the orbits of $N^{(012\bar{3}2)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(012\bar{3}2)}$. The orbits of $N^{(012\bar{3}2)}$ are $\{0, \bar{1}\}, \{1, \bar{0}\}, \{2, \bar{2}\}, \{3, \bar{3}\}$, and $\{4, \bar{4}\}$. The number of single cosets in the double

coset $[012\bar{3}2]$ is obtained by dividing $\frac{|N|}{|N^{(012\bar{3}2)}|} = \frac{120}{2} = 60$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[012\bar{3}2] = Nt_0t_1t_2\bar{t}_3t_2N = \{N\,(t_0t_1t_2\bar{t}_3t_2)^n\,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(012\bar{3}2)}$ and apply it to $Nt_0t_1t_2\bar{t}_3t_2$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_3t_2 \cdot t_0 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of 0 is of length two,
two t_i's will map the cayley graph from $[012\bar{3}2]$ to $[01230]$.

$Nt_0t_1t_2\bar{t}_3t_2 \cdot t_1 = Nt_0t_1t_2\bar{t}_0 \in [012\bar{0}]$.
Since the orbit of 1 is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}2]$ to $[012\bar{0}]$.

$Nt_0t_1t_2\bar{t}_3t_2 \cdot t_2 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of 2 is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}2]$ to $[012\bar{3}]$.

$Nt_0t_1t_2\bar{t}_3t_2 \cdot t_3 = Nt_0t_1t_2\bar{t}_4 \in [012\bar{4}]$.
Since the orbit of 3 is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}2]$ to $[012\bar{4}]$.

$Nt_0t_1t_2\bar{t}_3t_2 \cdot t_4 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [01230\bar{4}]$.
Since the orbit of 4 is of length one,
one t_i will expand the cayley graph from $[012\bar{3}2]$ to $[01230\bar{4}]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_3t_4N$ which we denoted by $[012\bar{3}4]$. To obtain the elements in $[012\bar{3}4]$, we must first find the point stabiliser of $0, 1, 2,$ and $\bar{3}$, $N^{012\bar{3}4}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, \bar{3}$, and 4 hence $\left|N^{012\bar{3}4}\right| = 1$. Since there are no additional relations which would increase these permutaions, we get that the set stabiliser $N^{(012\bar{3}4)} = N^{012\bar{3}4}$. Therefore, $\left|N^{(012\bar{3}4)}\right| = 1$.To find the orbits of $N^{(012\bar{3}4)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$

by $N^{(012\bar{3}4)}$. The orbits of $N^{(012\bar{3}4)}$ are $\{0\},\{1\},\{2\},\{3\},\{4\},\{\bar{0}\},\{\bar{1}\},\{\bar{2}\},\{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[012\bar{3}4]$ is obtained by dividing $\frac{|N|}{|N^{(012\bar{3}4)}|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012\bar{3}4] = Nt_0t_1t_2\bar{t}_3t_4N = \{N(t_0t_1t_2\bar{t}_3t_4)^n \,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(012\bar{3}4)}$ and apply it to $Nt_0t_1t_2\bar{t}_3t_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4 \in [0\bar{1}2\bar{3}4]$.
Since the orbit of 0 is of length one,
one t_i will map the cayley graph from $[012\bar{3}4]$ to $[0\bar{1}2\bar{3}4]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot t_1 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of 1 is of length one,
one t_i will collapse the cayley graph from $[012\bar{3}4]$ to $[0\bar{1}2\bar{4}]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot t_2 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of 2 is of length one,
one t_i will map the cayley graph from $[012\bar{3}4]$ to itself, $[012\bar{3}4]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot t_3 = Nt_0t_1t_2t_3\bar{t}_4 \in [0123\bar{4}]$.
Since the orbit of 3 is of length one,
one t_i will map the cayley graph from $[012\bar{3}4]$ to $[0123\bar{4}]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot t_4 = Nt_0t_1t_2t_3\bar{t}_4 \in [0123\bar{4}]$.
Since the orbit of 4 is of length one,
one t_i will map the cayley graph from $[012\bar{3}4]$ to $[0123\bar{4}]$.
Now we have two t_i's that will map the cayley graph
from $[012\bar{3}4]$ to $[0123\bar{4}]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot \bar{t}_0 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0$.
This is a new double coset which we will label as $[012\bar{3}4\bar{0}]$.

Since the orbit of $\bar{0}$ is of length one,
one t_i will extend the cayley graph from $[012\bar{3}4]$ to $[012\bar{3}4\bar{0}]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the cayley graph from $[012\bar{3}4]$ to $[0\bar{1}2\bar{4}3]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2$.
This is a new double coset which we will label as $[012\bar{3}4\bar{2}]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will extend the cayley graph from $[012\bar{3}4]$ to $[012\bar{3}4\bar{2}]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot \bar{t}_3 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [01230\bar{4}]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will extend the cayley graph from $[012\bar{3}4]$ to $[01230\bar{4}]$.

$Nt_0t_1t_2\bar{t}_3t_4 \cdot \bar{t}_4 = Nt_0t_1t_2\bar{t}_3 \in [012\bar{3}]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will collapse the cayley graph from $[012\bar{3}4]$ to $[012\bar{3}]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_4t_0N$ which we denoted by $[012\bar{4}0]$. To obtain the elements in $[012\bar{4}0]$, we must first find the point stabiliser of $0, 1, 2$, and $\bar{4}$, $N^{012\bar{4}0}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, \bar{4}$ and permute 3, hence $\left|N^{012\bar{4}0}\right| = 1$. Using magma, we obtain the relations $012\bar{4}0 \sim 410\bar{2}4 \sim 214\bar{0}2$. Using these relations, we can obtain the elements in the set stabiliser. These relations can be represented as permutations in S_5. These permutaions are $(042)\,(\bar{0}\bar{4}\bar{2})$ and $(024)\,(\bar{0}\bar{2}\bar{4})$. If we conjugate $Nt_0t_1t_2\bar{t}_4t_0$ by $(042)\,(\bar{0}\bar{4}\bar{2})$ we obtain $N\left(t_0t_1t_2\bar{t}_4t_0\right)^{(042)(\bar{0}\bar{4}\bar{2})\cdot} = Nt_4t_1t_0\bar{t}_2t_4$, but the relation tells us that $410\bar{2}4 \sim 012\bar{4}0$
$\Rightarrow Nt_4t_1t_0\bar{t}_2t_4 = Nt_0t_1t_2\bar{t}_4t_0$
$\Rightarrow (042)\,(\bar{0}\bar{4}\bar{2}) \in N^{(012\bar{4}0)}$
So, $N^{(012\bar{4}0)} \geq \left\langle (042)\,(\bar{0}\bar{4}\bar{2}), N^{012\bar{4}0} \right\rangle \Rightarrow \left|N^{(012\bar{4}0)}\right| = 3$. To find the orbits of $N^{(012\bar{4}0)}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(012\bar{4}0)}$. The orbits of $N^{(012\bar{4}0)}$ are $\{0, 2, 4\}, \{1\}, \{3\}, \{\bar{0}, \bar{2}, \bar{4}\}$,

$\{\bar{1}\}$, and $\{\bar{3}\}$. The number of single cosets in the double coset $[012\bar{4}0]$ is obtained by dividing $\frac{|N|}{|N^{(012\bar{4}0)}|} = \frac{120}{3} = 40$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012\bar{4}0] = Nt_0t_1t_2\bar{t}_4t_0N = \{N\,(t_0t_1t_2\bar{t}_4t_0)^n\,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(012\bar{4}0)}$ and apply it to $Nt_0t_1t_2\bar{t}_4t_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_4t_0 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$
Since the orbit of 0 is of length three,
three t_i's collapse the cayley graph from $[012\bar{4}0]$ to $[0\bar{1}2\bar{4}]$

$Nt_0t_1t_2\bar{t}_4t_0 \cdot t_1 = Nt_0\bar{t}_1t_2\bar{t}_0 \in [0\bar{1}2\bar{0}]$.
Since the orbit of 1 is of length one,
one t_i will map from $[012\bar{4}0]$ to $[0\bar{1}2\bar{0}]$.

$Nt_0t_1t_2\bar{t}_4t_0 \cdot t_3 = Nt_0t_1t_2t_3t_4\bar{t}_0 \in [01234\bar{0}]$.
Since the orbit of 3 is of length one,
one t_i will expand the cayley graph from $[012\bar{4}0]$ to $[01234\bar{0}]$.

$Nt_0t_1t_2\bar{t}_4t_0 \cdot \bar{t}_0 = Nt_0t_1t_2\bar{t}_4 \in [012\bar{4}]$.
Since the orbit of $\bar{0}$ is of length three,
three t_i's will colllapse the cayley graph from $[012\bar{4}0]$ to $[012\bar{4}]$.

$Nt_0t_1t_2\bar{t}_4t_0 \cdot \bar{t}_1 = Nt_0t_1t_2\bar{t}_3t_0 \in [012\bar{3}0]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{4}0]$ to $[012\bar{3}0]$.

$Nt_0t_1t_2\bar{t}_4t_0 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2 \in [012\bar{3}4\bar{2}]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will expand the Cayley graph from $[012\bar{4}0]$ to $[012\bar{3}4\bar{2}]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_4t_3N$ which we denoted by $[012\bar{4}3]$.

To obtain the elements in $[012\bar{4}3]$, we must first find the point stabiliser of $0, 1, 2, 3$, and $\bar{4}$, $N^{012\bar{4}0}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$ and $\bar{4}$, hence $\left|N^{012\bar{4}3}\right| = 1$. Since there are no additional relations which would increase these permutaions, we get that the set stabiliser $N^{(012\bar{4}3)} = N^{012\bar{4}3}$. Therefore, $\left|N^{(012\bar{4}3)}\right| = 1$.

To find the orbits of $N^{(012\bar{4}3)}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(012\bar{4}3)}$. The orbits of $N^{(012\bar{4}3)}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[012\bar{4}3]$ is obtained by dividing $\frac{|N|}{\left|N^{(012\bar{4}3)}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012\bar{4}3] = Nt_0t_1t_2\bar{t}_4t_3N = \{N\,(t_0t_1t_2\bar{t}_4t_3)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(012\bar{4}3)}$ and apply it to $Nt_0t_1t_2\bar{t}_4t_3$ to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1t_2\bar{t}_4t_3 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$
Since the orbit of 0 is of length three,
three t_i's collapse the cayley graph from $[012\bar{4}3]$ to $[0\bar{1}2\bar{4}3]$

$Nt_0t_1t_2\bar{t}_4t_3 \cdot t_1 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of 1 is of length one,
one t_i will map from $[012\bar{4}3]$ to $[0\bar{1}2\bar{3}]$.

$Nt_0t_1t_2\bar{t}_4t_3 \cdot t_2 = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of 2 is of length one,
one t_i's will map from $[012\bar{4}3]$ to $[012\bar{4}3]$.

$Nt_0t_1t_2\bar{t}_4t_3 \cdot t_3 = Nt_0t_1t_2\bar{t}_4\bar{t}_3 = Nt_0t_1t_2t_3t_4 \in [01234]$.
Since the orbit of 3 is of length one,
one t_i will expand the cayley graph from $[012\bar{4}3]$ to $[01234]$.

$Nt_0t_1t_2\bar{t}_4t_3 \cdot t_4 = Nt_0t_1t_2t_3t_4 \in [01234]$.
Since the orbit of 4 is of length one,
one t_i will expand the cayley graph from $[012\bar{4}3]$ to $[01234]$.

Thus, two things will map from $[012\bar{4}3]$ to $[01234]$.

$Nt_0t_1t_2\bar{t}_4t_3 \cdot \bar{t}_0 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \in [012\bar{3}4\bar{0}]$.
Since the orbit of $\bar{0}$ is of length three,
three t_i's will colllapse the cayley graph from $[012\bar{3}4\bar{0}]$ to $[012\bar{4}]$.

$Nt_0t_1t_2\bar{t}_4t_3 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_2\bar{t}_3t_4 \in [0\bar{1}2\bar{3}4]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{4}3]$ to $[0\bar{1}2\bar{3}4]$.

$Nt_0t_1t_2\bar{t}_4t_3 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4 \in [012\bar{3}0\bar{4}]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{4}3]$ to $[012\bar{3}0\bar{4}]$.

$Nt_0t_1t_2\bar{t}_4t_3 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_4 \in [012\bar{4}]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will expand the Cayley graph from $[012\bar{4}3]$ to $[012\bar{4}]$.

$Nt_0t_1t_2\bar{t}_4t_3 \cdot \bar{t}_4 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [01230\bar{4}]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will map the Cayley graph from $[012\bar{4}3]$ to $[01230\bar{4}]$.

Now we look at the double coset $Nt_0\bar{t}_1t_2\bar{t}_3t_0N$ which we denoted by $[0\bar{1}2\bar{3}0]$. To obtain the elements in $[0\bar{1}2\bar{3}0]$, we must first find the point stabiliser of $0, \bar{1}, 2$, and $\bar{3}$, $N^{0\bar{1}2\bar{3}0}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, \bar{1}, 2, \bar{3}$ and permute 4, hence $\left|N^{0\bar{1}2\bar{3}0}\right| = 1$. Using magma, we obtain the following relations $0\bar{1}2\bar{3}0 \sim \bar{3}0\bar{1}2\bar{3} \sim \bar{1}2\bar{3}0\bar{1} \sim 2\bar{3}0\bar{1}2$.
Using these relations, we can obtain the elements in the set stabiliser. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:
$0\bar{1}2\bar{3}0 \sim \bar{3}0\bar{1}2\bar{3} \Rightarrow (0\bar{3}2\bar{1})\,(\bar{0}3\bar{2}1)\,(4\bar{4})$
$0\bar{1}2\bar{3}0 \sim \bar{3}0\bar{1}2\bar{3} \Rightarrow (0\bar{1}2\bar{3})\,(\bar{0}1\bar{2}3)\,(4\bar{4})$
$0\bar{1}2\bar{3}0 \sim 2\bar{3}0\bar{1}2 \Rightarrow (02)\,(13)\,(\bar{0}\bar{2})\,(\bar{1}\bar{3})$

If we conjugate $Nt_0\bar{t}_1t_2\bar{t}_3t_0$ by the first permutation, $(0\bar{3}2\bar{1})(\bar{0}3\bar{2}1)(4\bar{4})$, we obtain the following: $N(t_0\bar{t}_1t_2\bar{t}_3t_0)^{(0\bar{3}2\bar{1})(\bar{0}3\bar{2}1)(4\bar{4})} = N\bar{t}_3t_0\bar{t}_1t_2\bar{t}_3$, but the relation tells us that $\bar{3}0\bar{1}2\bar{3} \sim 0\bar{1}2\bar{3}0$

$\Rightarrow N\bar{t}_3t_0\bar{t}_1t_2\bar{t}_3 = Nt_0\bar{t}_1t_2\bar{t}_3t_0$

$\Rightarrow (0\bar{3}2\bar{1})(\bar{0}3\bar{2}1)(4\bar{4}) \in N^{(0\bar{1}2\bar{3}0)}$

So, $N^{(0\bar{1}2\bar{3}0)} \geq \left\langle (0\bar{3}2\bar{1})(\bar{0}3\bar{2}1)(4\bar{4}), N^{0\bar{1}2\bar{3}0} \right\rangle \Rightarrow \left|N^{(0\bar{1}2\bar{3}0)}\right| = 4.$

To find the orbits of $N^{(0\bar{1}2\bar{3}0)}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(0\bar{1}2\bar{3}0)}$. The orbits of $N^{(0\bar{1}2\bar{3}0)}$ are $\{0, \bar{3}, 2, \bar{1}\}$, $\{\bar{0}, 3, \bar{2}, 1\}$, and $\{4, \bar{4}\}$. The number of single cosets in the double coset $[0\bar{1}2\bar{3}0]$ is obtained by dividing $\frac{|N|}{|N^{(0\bar{1}2\bar{3}0)}|} = \frac{120}{4} = 30$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[0\bar{1}2\bar{3}0] = Nt_0\bar{t}_1t_2\bar{t}_3t_0N = \{N(t_0\bar{t}_1t_2\bar{t}_3t_0)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(0\bar{1}2\bar{3}0)}$ and apply it to $Nt_0\bar{t}_1t_2\bar{t}_3t_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0\bar{t}_1t_2\bar{t}_3t_0 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_3\bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of 0 is of length four,
four t_i's will collapse the cayley graph from $[0\bar{1}2\bar{3}0]$ to $[0\bar{1}2\bar{4}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_0 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of $\bar{0}$ is of length four,
four t_i's will collapse the cayley graph from $[0\bar{1}2\bar{3}0]$ to $[0\bar{1}2\bar{3}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_0 \cdot t_4 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \in [012\bar{3}4\bar{0}]$.
Since the orbit of 4 is of length two,
two t_i's will expand the cayley graph from $[0\bar{1}2\bar{3}0]$ to $[012\bar{3}4\bar{0}]$

Now we look at the double coset $Nt_0\bar{t}_1t_2\bar{t}_3t_4N$ which we denoted by $[0\bar{1}2\bar{3}4]$. To obtain the elements in $[0\bar{1}2\bar{3}4]$, we must first find the point stabiliser of $0, \bar{1}, 2, \bar{3}$ and 4, $N^{0\bar{1}2\bar{3}4}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, \bar{1}, 2, \bar{3}$ and 4, hence $\left|N^{0\bar{1}2\bar{3}4}\right| = 1$. Since there are no additional relations which would increase these permutaions, we get that the set stabiliser $N^{(0\bar{1}2\bar{3}4)} = N^{0\bar{1}2\bar{3}4}$. Therefore, $\left|N^{(0\bar{1}2\bar{3}4)}\right| = 1$. To find the orbits of $N^{(0\bar{1}2\bar{3}4)}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(0\bar{1}2\bar{3}4)}$. The orbits of

$N^{(0\bar{1}2\bar{3}4)}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[0\bar{1}2\bar{3}4]$ is obtained by dividing $\frac{|N|}{|N^{(0\bar{1}2\bar{3}4)}|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[0\bar{1}2\bar{3}4] = Nt_0\bar{t}_1t_2\bar{t}_3t_4N = \{N(t_0\bar{t}_1t_2\bar{t}_3t_4)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(0\bar{1}2\bar{3}4)}$ and apply it to $Nt_0\bar{t}_1t_2\bar{t}_3t_4$ to determine if it will extend or collapse the Cayley graph:

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$.
Since the orbit of 0 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[0\bar{1}2\bar{4}3]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot t_1 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of 1 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[012\bar{3}4]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot t_2 = Nt_0t_1t_2t_3t_4 \in [01234]$.
Since the orbit of 2 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[01234]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot t_3 = Nt_0t_1t_2t_3\bar{t}_4 \in [0123\bar{4}]$.
Since the orbit of 3 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[0123\bar{4}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot t_4 = Nt_0\bar{t}_1t_2\bar{t}_3\bar{t}_4 = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of 4 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[012\bar{4}3]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0$.
This is a new double coset which we will label as $[0\bar{1}2\bar{3}4\bar{0}]$
Since the orbit of $\bar{0}$ is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[0\bar{1}2\bar{3}4\bar{0}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot \bar{t}_1 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \in [012\bar{3}4\bar{0}]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[012\bar{3}4\bar{0}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot \bar{t}_2 = Nt_0t_1t_2t_3t_4\bar{t}_0 \in [01234\bar{0}]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will expand the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[01234\bar{0}]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot \bar{t}_3 = Nt_0t_1t_2t_3t_0t_4 \in [012304]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i's will expand the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[012304]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4 \cdot \bar{t}_4 = Nt_0\bar{t}_1t_2\bar{t}_3 \in [0\bar{1}2\bar{3}]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will collapse the cayley graph from $[0\bar{1}2\bar{3}4]$ to $[0\bar{1}2\bar{3}]$.

Now we look at the double coset $Nt_0\bar{t}_1t_2\bar{t}_4t_3N$ which we denoted by $[0\bar{1}2\bar{4}3]$. To obtain the elements in $[0\bar{1}2\bar{4}3]$, we must first find the point stabiliser of $0, \bar{1}, 2, 3$ and $\bar{4}$, $N^{0\bar{1}2\bar{4}3}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, \bar{1}, 2, 3$ and $\bar{4}$, hence $\left|N^{0\bar{1}2\bar{4}3}\right| = 1$. Since there are no additional relations which would increase these permutaions, we get that the set stabiliser $N^{(0\bar{1}2\bar{4}3)} = N^{0\bar{1}2\bar{4}3}$. Therefore, $\left|N^{(0\bar{1}2\bar{4}3)}\right| = 1$. To find the orbits of $N^{(0\bar{1}2\bar{4}3)}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(0\bar{1}2\bar{4}3)}$. The orbits of $N^{(0\bar{1}2\bar{4}3)}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[0\bar{1}2\bar{4}3]$ is obtained by dividing $\frac{|N|}{\left|N^{(0\bar{1}2\bar{4}3)}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[0\bar{1}2\bar{4}3] = Nt_0\bar{t}_1t_2\bar{t}_4t_3N = \{N\left(t_0\bar{t}_1t_2\bar{t}_4t_3\right)^n | n \in N\}$

Next, we must take a representative of each orbit of $N^{(0\bar{1}2\bar{4}3)}$ and apply it to $Nt_0\bar{t}_1t_2\bar{t}_4t_3$ to determine if it will extend or collapse the Cayley graph:

$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4 \in [0\bar{1}2\bar{3}4]$.
Since the orbit of 0 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[0\bar{1}2\bar{3}4]$.

$Nt_0\bar{t}_1t_2\bar{t}_4t_{31} = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of 1 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[012\bar{4}3]$.

$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot t_2 = Nt_0t_1t_2t_3\bar{t}_4 \in [0123\bar{4}]$.
Since the orbit of 2 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[0123\bar{4}]$.

$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot t_3 = Nt_0\bar{t}_1t_2\bar{t}_4\bar{t}_3 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of 3 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[012\bar{3}4]$.

$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot t_4 = Nt_0t_1t_2t_3t_4 \in [01234]$.
Since the orbit of 4 is of length one,
one t_i will map the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[01234]$.
$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0 \in [0\bar{1}2\bar{3}4\bar{0}]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will extend the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[0\bar{1}2\bar{3}4\bar{0}]$.

$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot \bar{t}_1 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \in [012\bar{3}4\bar{0}]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will extend the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[012\bar{3}4\bar{0}]$.

$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot \bar{t}_2 = Nt_0t_1t_2t_3\bar{t}_4t_1 \in [0123\bar{4}1]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will extend the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[0123\bar{4}1]$.

$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot \bar{t}_3 = Nt_0\bar{t}_1t_2\bar{t}_4 \in [0\bar{1}2\bar{4}]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will collapse the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[0\bar{1}2\bar{4}]$.

$Nt_0\bar{t}_1t_2\bar{t}_4t_3 \cdot \bar{t}_4 = Nt_0t_1t_2t_3t_0t_4 \in [012304]$.

Since the orbit of $\bar{4}$ is of length one,

one t_i will extend the cayley graph from $[0\bar{1}2\bar{4}3]$ to $[012304]$.

Now we look at the double coset $Nt_0t_1t_2t_3t_0t_1N$ which we denoted by $[012301]$. To obtain the elements in $[012301]$, we must first find the point stabiliser of $0, 1, 2$, and 3, N^{012301}. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$ and permute 4, hence $\left|N^{012301}\right| = 1$. Using magma we obtain the following relations:
$012301 \sim 230123 \sim \bar{2}\bar{1}\bar{0}\bar{3}\bar{2}\bar{1} \sim \bar{0}\bar{3}\bar{2}\bar{1}\bar{0}\bar{3} \sim \bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2} \sim 103210 \sim \bar{3}\bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \sim 321032$.
Using these relations, we can obtain the elements in the set stabiliser. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:
$012301 \sim 230123 \Rightarrow (02)\,(13)\,(4\bar{4})\,(\bar{0}\bar{2})\,(\bar{1}\bar{3})$
$012301 \sim \bar{3}\bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \Rightarrow (1\bar{0}3\bar{2})\,(\bar{1}0\bar{3}2)\,(4\bar{4})$
If we conjugate $Nt_0t_1t_2t_3t_0t_1$ by the first permutation, $(02)\,(13)\,(4\bar{4})\,(\bar{0}\bar{2})\,(\bar{1}\bar{3})$, we obtain the following: $N\left(t_0t_1t_2t_3t_0t_1\right)^{(02)(13)(4\bar{4})(\bar{0}\bar{2})(\bar{1}\bar{3})} = Nt_2t_3t_0t_1t_2t_3$, but the relation tells us that $230123 \sim 012301$
$\Rightarrow Nt_2t_3t_0t_1t_2t_3 = Nt_0t_1t_2t_3t_0t_1$
$\Rightarrow (02)\,(13)\,(4\bar{4})\,(\bar{0}\bar{2})\,(\bar{1}\bar{3}) \in N^{(012301)}$
So, $N^{(012301)} \geq \left\langle (02)\,(13)\,(4\bar{4})\,(\bar{0}\bar{2})\,(\bar{1}\bar{3}), N^{012301} \right\rangle \Rightarrow \left|N^{(012301)}\right| \geq 2$.
If we conjugate $Nt_0t_1t_2t_3t_0t_1$ by the second permutation, $(1\bar{0}3\bar{2})\,(\bar{1}0\bar{3}2)\,(4\bar{4})$, we obtain the following: $N\left(t_0t_1t_2t_3t_0t_1\right)^{(1\bar{0}3\bar{2})(\bar{1}0\bar{3}2)(4\bar{4})} = N\bar{t}_3\bar{t}_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_0$, but the relation tells us that $\bar{3}\bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \sim 012301$
$\Rightarrow N\bar{t}_3\bar{t}_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_0 = Nt_0t_1t_2t_3t_0t_1$
$\Rightarrow (1\bar{0}3\bar{2})\,(\bar{1}0\bar{3}2)\,(4\bar{4}) \in N^{(012301)}$
So, $N^{(012301)} = \left\langle (1\bar{0}3\bar{2})\,(\bar{1}0\bar{3}2)\,(4\bar{4}), (02)\,(13)\,(4\bar{4})\,(\bar{0}\bar{2})\,(\bar{1}\bar{3}), N^{012301} \right\rangle \Rightarrow \left|N^{(012301)}\right| = 8$.
To find the orbits of $N^{(012301)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(012301)}$. The orbits of $N^{(012301)}$ are $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$, and $\{4, \bar{4}\}$. The number of single cosets in the double coset $[012301]$ is obtained by dividing $\frac{|N|}{\left|N^{(012301)}\right|} = \frac{120}{8} = 15$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[012301] = Nt_0t_1t_2t_3t_0t_1N = \{N\left(t_0t_1t_2t_3t_0t_1\right)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(012301)}$ and apply it to $Nt_0t_1t_2t_3t_0t_1$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3t_0t_1 \cdot \bar{t}_1 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of $\bar{1}$ is of length eight,
eight t_i's will collapse the cayley graph from [012301] to [01230].

$Nt_0t_1t_2t_3t_0t_1 \cdot t_4 = Nt_0t_1t_2t_3t_0t_1t_4$.
This is a new double coset which we will label as [0123014].
Since the orbit of 4 is of length two,
two t_i's will expand the cayley graph from [012301] to [0123014].

Now we look at the double coset $Nt_0t_1t_2t_3t_0t_4N$ which we denoted by [012304]. To obtain the elements in [012304], we must first find the point stabiliser of $0, 1, 2, 3$ and 4, N^{012304}. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$ and 4, hence $\left|N^{01234}\right| = 1$. Since there are no additional relations which would increase these permutations, we obtain that the set stabiliser $N^{(012304)} = N^{012304}$, and so $\left|N^{(012304)}\right| = 1$. To find the orbits of $N^{(012304)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(012304)}$. The orbits of $N^{(012304)}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$, and $\{\bar{4}\}$.The number of single cosets in the double coset [012304] is obtained by dividing $\frac{|N|}{\left|N^{(012304)}\right|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[012304] = Nt_0t_1t_2t_3t_0t_4N = \{N(t_0t_1t_2t_3t_0t_4)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(012304)}$ and apply it to $Nt_0t_1t_2t_3t_0t_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3t_0t_4 \cdot t_0 = Nt_0t_1t_2t_3\bar{t}_4 \in [0123\bar{4}]$.
Since the orbit of 0 is of length one,
one t_i will collapse the cayley graph from [012304] to $[0123\bar{4}]$.

$Nt_0t_1t_2t_3t_0t_4 \cdot t_1 = Nt_0t_1t_2t_3t_0t_4 \in [012304]$.
Since the orbit of 1 is of length one,
one t_i will map the cayley graph from [012304] to [012304].

$Nt_0t_1t_2t_3t_0t_4 \cdot t_2 = Nt_0t_1t_2t_3t_4 \in [01234]$.
Since the orbit of 2 is of length one,
one t_i will collapse the cayley graph from [012304] to [01234].

$Nt_0t_1t_2t_3t_0t_4 \cdot t_3 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [012304]$.
Since the orbit of 3 is of length one,
one t_i will map the cayley graph from [012304] to [012304].

$Nt_0t_1t_2t_3t_0t_4 \cdot t_4 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [012304]$.
Since the orbit of 4 is of length one,
one t_i will map the cayley graph from [012304] to [012304].
Now, two t_i's will map the cayley graph from [012304] to [012304].

$Nt_0t_1t_2t_3t_0t_4 \cdot \bar{t}_0 = Nt_0t_1t_2\bar{t}_3t_4t_0 \in [012\bar{3}40]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will map the cayley graph from [012304] to [012340].

$Nt_0t_1t_2t_3t_0t_4 \cdot \bar{t}_1 = Nt_0t_1t_2t_3t_0t_4\bar{t}_1$.
This is a new double coset which we will label as [012304$\bar{1}$].
Since the orbit of $\bar{1}$ is of length one,
one t_i' will extend the cayley graph from [012304] to [012304$\bar{1}$].

$Nt_0t_1t_2t_3t_0t_4 \cdot \bar{t}_2 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will collapse the cayley graph from [012304] to [0$\bar{1}$2$\bar{4}$3].

$Nt_0t_1t_2t_3t_0t_4 \cdot \bar{t}_3 = Nt_0t_1t_2t_3t_0t_1t_4 \in [0123014]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will extend the cayley graph from [012304] to [0123014].

$Nt_0t_1t_2t_3t_0t_4 \cdot \bar{t}_4 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of $\bar{4}$ is of length one,

one t_i will collapse the cayley graph from [012304] to [01230].

Now we look at the double coset $Nt_0t_1t_2t_3t_0\bar{t}_4N$ which we denoted by $[01230\bar{4}]$. To obtain the elements in $[01230\bar{4}]$, we must first find the point stabiliser of $0,1,2,3$ and $\bar{4}$, $N^{01230\bar{4}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0,1,2,3$ and $\bar{4}$, hence $\left|N^{0123\bar{4}}\right| = 1$. Since there are no additional relations which would increase these permutations, we obtain that the set stabiliser $N^{(01230\bar{4})} = N^{01230\bar{4}}$, and so $\left|N^{(01230\bar{4})}\right| = 1$. To find the orbits of $N^{(01230\bar{4})}$ we conjugate $0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3}$, and $\bar{4}$ by $N^{(01230\bar{4})}$. The orbits of $N^{(01230\bar{4})}$ are $\{0\},\{1\},\{2\},\{3\},\{4\},\{\bar{0}\},\{\bar{1}\},\{\bar{2}\},\{\bar{3}\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset $[01230\bar{4}]$ is obtained by dividing $\frac{|N|}{|N^{(01230\bar{4})}|} = \frac{120}{1} = 120$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012304] = Nt_0t_1t_2t_3t_0\bar{t}_4N = \{N\,(t_0t_1t_2t_3t_0\bar{t}_4)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(01230\bar{4})}$ and apply it to $Nt_0t_1t_2t_3t_0\bar{t}_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot t_0 = Nt_0t_1t_2t_3t_4 \in [01234]$.
Since the orbit of 0 is of length one,
one t_i will collapse the cayley graph from $[01230\bar{4}]$ to [01234].

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot t_1 = Nt_0t_1t_2t_3t_0t_4 \in [012304]$.
Since the orbit of 1 is of length one,
one t_i will map the cayley graph from $[01230\bar{4}]$ to [012304].

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot t_2 = Nt_0t_1t_2t_3\bar{t}_4 \in [0123\bar{4}]$.
Since the orbit of 2 is of length one,
one t_i will collapse the cayley graph from $[01230\bar{4}]$ to $[0123\bar{4}]$.

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot t_3 = Nt_0t_1t_2t_3t_0\bar{t}_4 \in [01230\bar{4}]$.
Since the orbit of 3 is of length one,
one t_i will map the cayley graph from $[01230\bar{4}]$ to $[01230\bar{4}]$.

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot t_4 = Nt_0t_1t_2t_3t_0 \in [01230]$.
Since the orbit of 4 is of length one,
one t_i will collapse the cayley graph from $[01230\bar{4}]$ to [01230].

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot \bar{t}_0 = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will collapse the cayley graph from $[01230\bar{4}]$ to $[012\bar{4}3]$.

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot \bar{t}_1 = Nt_0t_1t_2t_3t_0t_1t_4 \in [0123014]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will extend the cayley graph from $[01230\bar{4}]$ to [0123014].

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will collapse the cayley graph from $[01230\bar{4}]$ to $[012\bar{3}4]$.

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_3t_2 \in [012\bar{3}2]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will collapse the cayley graph from $[01230\bar{4}]$ to $[012\bar{3}2]$.

$Nt_0t_1t_2t_3t_0\bar{t}_4 \cdot \bar{t}_4 = Nt_0t_1t_2t_3t_0t_4 \in [012304]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will map the cayley graph from $[01230\bar{4}]$ to [012304].
Now, two t_i's map the cayley graph from $[01230\bar{4}]$ to [012304].

Now we look at the double coset $Nt_0t_1t_2t_3t_4\bar{t}_0N$ which we denoted by $[01234\bar{0}]$. To obtain the elements in $[01234\bar{0}]$, we must first find the point stabiliser of $0, 1, 2, 3, 4$ and $\bar{0}$, $N^{01234\bar{0}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3, 4, \bar{0}$, hence $\left|N^{01234\bar{0}}\right| = 1$. Using magma we obtain the following relations:
$01234\bar{0} \sim 41032\bar{4} \sim 21435\bar{2}$.
Using these relations, we can obtain the elements in the set stabiliser, $N^{(01234\bar{0})}$. These relations can be represented as permutations in S_5. These permutaions are obtained as

follows:

$012340 \sim 41032\bar{4} \Rightarrow (042)(\bar{0}\bar{4}\bar{2})$

$012340 \sim 21435\bar{2} \Rightarrow (024)(\bar{0}\bar{2}\bar{4})$

If we conjugate $Nt_0t_1t_2t_3t_4\bar{t}_0$ by the first permutation, $(042)(\bar{0}\bar{4}\bar{2})$, we obtain the following: $N(t_0t_1t_2t_3t_4\bar{t}_0)^{(042)(\bar{0}\bar{4}\bar{2})} = Nt_4t_1t_0t_3t_2\bar{t}_4$, but the relation tells us that $41032\bar{4} \sim 012340$

$\Rightarrow Nt_4t_1t_0t_3t_2\bar{t}_4 = Nt_0t_1t_2t_3t_4\bar{t}_0$

$\Rightarrow (042)(\bar{0}\bar{4}\bar{2}) \in N^{(012340)}$

So, $N^{(012340)} \geq \left\langle (042)(\bar{0}\bar{4}\bar{2}), N^{012340} \right\rangle \Rightarrow \left|N^{(012340)}\right| \geq 3$.

Hence $\left|N^{(012340)}\right| = 3$. To find the orbits of $N^{(012340)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(012340)}$. The orbits of $N^{(012340)}$ are $\{0,2,4\}, \{1\}, \{3\}, \{\bar{0},\bar{2},\bar{4}\}, \{\bar{1}\}$, and $\{\bar{3}\}$. The number of single cosets in the double coset $[012340]$ is obtained by dividing $\frac{|N|}{\left|N^{(012340)}\right|} = \frac{120}{3} = 40$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012340] = Nt_0t_1t_2t_3t_4\bar{t}_0N = \{N(t_0t_1t_2t_3t_4\bar{t}_0)^n \,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(012340)}$ and apply it to $Nt_0t_1t_2t_3t_4\bar{t}_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3t_4\bar{t}_0 \cdot t_0 = Nt_0t_1t_2t_3t_4 \in [01234]$.
Since the orbit of 0 is of length three,
three t_i's will collapse the cayley graph from $[012340]$ to $[01234]$.

$Nt_0t_1t_2t_3t_4\bar{t}_0 \cdot t_1 = Nt_0t_1t_2t_3\bar{t}_4t_1 \in [0123\bar{4}1]$
Since the orbit of 1 is of length one,
one t_i will map the cayley graph from $[012340]$ to $[0123\bar{4}1]$.

$Nt_0t_1t_2t_3t_4\bar{t}_0 \cdot t_3 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2 \in [012\bar{3}4\bar{2}]$.
Since the orbit of 3 is of length one,
one t_i will map the cayley graph from $[012340]$ to $[012\bar{3}4\bar{2}]$.

$Nt_0t_1t_2t_3t_4\bar{t}_0 \cdot \bar{t}_0 = Nt_0t_1t_2t_3t_4t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4 \in [0\bar{1}2\bar{3}4]$.
Since the orbit of $\bar{0}$ is of length three,

three t_i's will collapse the cayley graph from $[012340]$ to $[0\bar{1}2\bar{3}4]$

$Nt_0t_1t_2t_3t_4\bar{t}_0 \cdot \bar{t}_1 = Nt_0t_1t_2t_3t_0t_4\bar{t}_1 \in [012304\bar{1}]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will extend the cayley graph from $[012340]$ to $[012304\bar{1}]$.

$Nt_0t_1t_2t_3t_4\bar{t}_0 \cdot \bar{t}_3 = Nt_0t_1t_2\bar{t}_4t_0 \in [012\bar{4}0]$.
Since the orbit of $\bar{3}$ is of length one,
one t_i will collapse the cayley graph from $[012340]$ to $[012\bar{4}0]$.

Now we look at the double coset $Nt_0t_1t_2t_3\bar{t}_4t_1N$ which we denoted by $[0123\bar{4}1]$. To obtain the elements in $[0123\bar{4}1]$, we must first find the point stabiliser of $0, 1, 2, 3$ and $\bar{4}$, $N^{0123\bar{4}1}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$, and permute $\bar{4}$ hence $\left|N^{0123\bar{4}1}\right| = 1$. Using magma we obtain the following relations: $0123\bar{4}1 \sim 0324\bar{1}3 \sim 0421\bar{3}4$.
Using these relations, we can obtain the elements in the set stabiliser, $N^{(0123\bar{4}1)}$. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:
$0123\bar{4}1 \sim 0324\bar{1}3 \Rightarrow (134)(\bar{1}\bar{3}\bar{4})$
$0123\bar{4}1 \sim 0421\bar{3}4 \Rightarrow (143)(\bar{1}\bar{4}\bar{3})$
If we conjugate $Nt_0t_1t_2t_3\bar{t}_4t_1$ by the first permutation, $(134)(\bar{1}\bar{3}\bar{4})$, we obtain the following: $N(t_0t_1t_2t_3\bar{t}_4t_1)^{(134)(\bar{1}\bar{3}\bar{4})} = Nt_0t_3t_2t_4\bar{t}_1t_3$, but the relation tells us that $0324\bar{1}3 \sim 0123\bar{4}1$
$\Rightarrow Nt_0t_3t_2t_4\bar{t}_1t_3 = Nt_0t_1t_2t_3\bar{t}_4t_1$
$\Rightarrow (134)(\bar{1}\bar{3}\bar{4}) \in N^{(0123\bar{4}1)}$
So, $N^{(0123\bar{4}1)} \geq \left\langle (134)(\bar{1}\bar{3}\bar{4}), N^{0123\bar{4}1} \right\rangle \Rightarrow \left|N^{(0123\bar{4}1)}\right| \geq 3$.
Hence $\left|N^{(0123\bar{4}1)}\right| = 3$. To find the orbits of $N^{(0123\bar{4}1)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(0123\bar{4}1)}$. The orbits of $N^{(0123\bar{4}1)}$ are $\{0\}, \{2\}, \{1, 3, 4\}, \{\bar{0}\}, \{\bar{2}\}$, and $\{\bar{1}, \bar{3}, \bar{4}\}$. The number of single cosets in the double coset $[0123\bar{4}]$ is obtained by dividing $\frac{|N|}{\left|N^{(0123\bar{4}1)}\right|} = \frac{120}{3} = 40$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[0123\bar{4}1] = Nt_0t_1t_2t_3\bar{t}_4t_1N = \{N(t_0t_1t_2t_3\bar{t}_4t_1)^n \,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(012\bar{3}\bar{4}1)}$ and apply it to $Nt_0t_1t_2t_3\bar{t}_4t_1$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3\bar{t}_4t_1 \cdot t_0 = Nt_0t_1t_2t_3t_0t_4\bar{t}_1 \in [012304\bar{1}]$.
Since the orbit of 0 is of length one,
one t_i will extend the cayley graph from $[012\bar{3}\bar{4}1]$ to $[012304\bar{1}]$.

$Nt_0t_1t_2t_3\bar{t}_4t_1 \cdot t_1 = Nt_0t_1t_2t_3\bar{t}_4\bar{t}_1 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$
Since the orbit of 1 is of length three,
three t_i's will collapse the cayley graph from $[012\bar{3}\bar{4}1]$ to $[0\bar{1}2\bar{4}3]$

$Nt_0t_1t_2t_3\bar{t}_4t_1 \cdot t_2 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of 2 is of length one,
one t_i will map the cayley graph from $[012\bar{3}\bar{4}1]$ to $[012\bar{3}4]$.

$Nt_0t_1t_2t_3\bar{t}_4t_1 \cdot \bar{t}_0 = Nt_0t_1t_2t_3t_4\bar{t}_0 \in [01234\bar{0}]$.
Since the orbit of $\bar{0}$ is of length one,
one t_i will map the cayley graph from $[012\bar{3}\bar{4}1]$ to $[01234\bar{0}]$

$Nt_0t_1t_2t_3\bar{t}_4t_1 \cdot \bar{t}_1 = Nt_0t_1t_2t_3\bar{t}_4 \in [0123\bar{4}]$.
Since the orbit of $\bar{1}$ is of length three,
three t_i's will collapse the cayley graph from $[012\bar{3}\bar{4}1]$ to $[0123\bar{4}]$.

$Nt_0t_1t_2t_3\bar{t}_4t_1 \cdot \bar{t}_2 = Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4 \in [012\bar{3}0\bar{4}]$.
Since the orbit of $\bar{2}$ is of length one,
one t_i will map the cayley graph from $[012\bar{3}\bar{4}1]$ to $[012\bar{3}0\bar{4}]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4N$ which we denoted by $[012\bar{3}0\bar{4}]$. To obtain the elements in $[012\bar{3}0\bar{4}]$, we must first find the point stabiliser of $0, 1, 2, \bar{3}$ and $\bar{4}$, $N^{012\bar{3}0\bar{4}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, \bar{3}$, and $\bar{4}$ hence $\left|N^{012\bar{3}0\bar{4}}\right| = 1$. Using magma, we obtain the relations $012\bar{3}0\bar{4} \sim 310\bar{2}3\bar{4} \sim 213\bar{0}2\bar{4} \sim \bar{2}\bar{4}\bar{3}0\bar{2}1 \sim \bar{3}\bar{4}\bar{0}2\bar{3}1 \sim \bar{0}\bar{4}\bar{2}3\bar{0}1$. Using these relations, we can obtain the elements

in the set stabiliser, $N^{(012\bar{3}0\bar{4})}$. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:

$012\bar{3}0\bar{4} \sim \bar{2}\bar{4}\bar{3}0\bar{2}1 \Rightarrow (1\bar{4})\,(2\bar{3}0\bar{2}3\bar{0})\,(\bar{1}4)$

If we conjugate $Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4$ by this permutation, $(1\bar{4})\,(2\bar{3}0\bar{2}3\bar{0})\,(\bar{1}4)$, we obtain the following: $N\left(t_0t_1t_2\bar{t}_3t_0\bar{t}_4\right)^{(1\bar{4})(2\bar{3}0\bar{2}3\bar{0})(\bar{1}4)} = N\bar{t}_2\bar{t}_4\bar{t}_3t_0\bar{t}_2t_1$, but the relation tells us that $\bar{2}\bar{4}\bar{3}0\bar{2}1 \sim 012\bar{3}0\bar{4}$

$\Rightarrow N\bar{t}_2\bar{t}_4\bar{t}_3t_0\bar{t}_2t_1 = Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4$

$\Rightarrow (1\bar{4})\,(2\bar{3}0\bar{2}3\bar{0})\,(\bar{1}4) \in N^{(012\bar{3}0\bar{4})}$

So, $N^{(012\bar{3}0\bar{4})} \geq \left\langle (1\bar{4})\,(2\bar{3}0\bar{2}3\bar{0})\,(\bar{1}4), N^{012\bar{3}0\bar{4}} \right\rangle \Rightarrow \left|N^{(012\bar{3}0\bar{4})}\right| \geq 6.$

Hence $\left|N^{(012\bar{3}0\bar{4})}\right| = 6.$

To find the orbits of $N^{(012\bar{3}0\bar{4})}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(012\bar{3}0\bar{4})}$. The orbits of $N^{(012\bar{3}0\bar{4})}$ are $\{1, \bar{4}\}, \{\bar{1}, 4\}$, and $\{0, 2, 3, \bar{0}, \bar{2}, \bar{3}\}$. The number of single cosets in the double coset $[012\bar{3}0\bar{4}]$ is obtained by dividing $\frac{|N|}{\left|N^{(012\bar{3}0\bar{4})}\right|} = \frac{120}{6} = 20$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012\bar{3}0\bar{4}] = Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4N = \{N\left(t_0t_1t_2\bar{t}_3t_0\bar{t}_4\right)^n | n \in N\}.$

Next, we must take a representative of each orbit of $N^{(012\bar{3}0\bar{4})}$ and apply it to $Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4 \cdot t_0 = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of 0 is of length six,
six t_i's will collapse the cayley graph from $[012\bar{3}0\bar{4}]$ to $[012\bar{4}3]$.

$Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4 \cdot t_1 = Nt_0t_1t_2t_3\bar{t}_4t_1 \in [0123\bar{4}1]$.
Since the orbit of 1 is of length two,
two t_i's will map the cayley graph from $[012\bar{3}0\bar{4}]$ to $[0123\bar{4}1]$.

$Nt_0t_1t_2\bar{t}_3t_0\bar{t}_4 \cdot t_4 = Nt_0t_1t_2\bar{t}_3t_0 \in [012\bar{3}0]$.
Since the orbit of 4 is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}0\bar{4}]$ to $[012\bar{3}0]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0N$ which we denoted by $[012\bar{3}4\bar{0}]$. To obtain the elements in $[012\bar{3}4\bar{0}]$, we must first find the point stabiliser of $0, 1, 2, 3$ and

4, $N^{012\bar{3}4\bar{0}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, \bar{3}$, and 4 hence $\left|N^{012\bar{3}4\bar{0}}\right| = 1$. Using magma, we obtain the relation $012\bar{3}4\bar{0} \sim 31\bar{4}0\bar{2}\bar{3}$. Using this relation, we can obtain the elements in the set stabiliser, $N^{(012\bar{3}4\bar{0})}$. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:

$012\bar{3}4\bar{0} \sim 31\bar{4}0\bar{2}\bar{3} \Rightarrow (03)(24)(\bar{0}\bar{3})(\bar{2}\bar{4})$

If we conjugate $Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0$ by this permutation, $(03)(24)(\bar{0}\bar{3})(\bar{2}\bar{4})$, we obtain the following: $N(t_0t_1t_2\bar{t}_3t_4\bar{t}_0)^{(03)(24)(\bar{0}\bar{3})(\bar{2}\bar{4})} = Nt_3t_1t_4\bar{t}_0t_2\bar{t}_3$, but the relation tells us that $31\bar{4}0\bar{2}\bar{3} \sim 012\bar{3}4\bar{0}$

$\Rightarrow Nt_3t_1t_4\bar{t}_0t_2\bar{t}_3 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0$

$\Rightarrow (03)(24)(\bar{0}\bar{3})(\bar{2}\bar{4}) \in N^{(012\bar{3}4\bar{0})}$

So, $N^{(012\bar{3}4\bar{0})} \geq \left\langle (03)(24)(\bar{0}\bar{3})(\bar{2}\bar{4}), N^{012\bar{3}4\bar{0}} \right\rangle \Rightarrow \left|N^{(012\bar{3}4\bar{0})}\right| \geq 2$.

Hence $\left|N^{(012\bar{3}4\bar{0})}\right| = 2$. To find the orbits of $N^{(012\bar{3}4\bar{0})}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(012\bar{3}4\bar{0})}$. The orbits of $N^{(012\bar{3}4\bar{0})}$ are $\{1\}, \{2,4\}, \{0,3\}, \{\bar{1}\}, \{\bar{2},\bar{4}\}$, and $\{\bar{0},\bar{3}\}$. The number of single cosets in the double coset $[012\bar{3}4\bar{0}]$ is obtained by dividing $\frac{|N|}{\left|N^{(012\bar{3}4\bar{0})}\right|} = \frac{120}{2} = 60$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012\bar{3}4\bar{0}] = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0N = \{N(t_0t_1t_2\bar{t}_3t_4\bar{t}_0)^n \,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(012\bar{3}4\bar{0})}$ and apply it to $Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \cdot t_0 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of 0 is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}4\bar{0}]$ to $[012\bar{3}4]$.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \cdot t_1 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \in [012\bar{3}4\bar{0}]$.
Since the orbit of 1 is of length one,
one t_i will map the cayley graph from $[012\bar{3}4\bar{0}]$ to itself , $[012\bar{3}4\bar{0}]$.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \cdot t_2 = Nt_0t_1t_2\bar{t}_4t_3 \in [012\bar{4}3]$.
Since the orbit of 2 is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}4\bar{0}]$ to $[012\bar{4}3]$.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \cdot \bar{t}_0 = Nt_0t_1t_2\bar{t}_3t_4t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4 \in [0\bar{1}2\bar{3}4]$.
Since the orbit of $\bar{0}$ is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}4\bar{0}]$ to $[0\bar{1}2\bar{3}4]$.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \cdot \bar{t}_1 = Nt_0\bar{t}_1t_2\bar{t}_3t_0 \in [0\bar{1}2\bar{3}0]$.
Since the orbit of $\bar{1}$ is of length one,
one t_i will collapse the cayley graph from $[012\bar{3}4\bar{0}]$ to $[0\bar{1}2\bar{3}0]$.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_0 \cdot \bar{t}_2 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$.
Since the orbit of $\bar{2}$ is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}4\bar{0}]$ to $[0\bar{1}2\bar{4}3]$.

Now we look at the double coset $Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2N$ which we denoted by $[012\bar{3}4\bar{2}]$. To obtain the elements in $[012\bar{3}4\bar{2}]$, we must first find the point stabiliser of $0, 1, 2, 3$ and 4, $N^{012\bar{3}4\bar{2}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, \bar{3}$, and 4 hence $\left|N^{012\bar{3}4\bar{2}}\right| = 1$. Using magma, we obtain the relations $012\bar{3}4\bar{2} \sim 014\bar{2}3\bar{4} \sim \bar{1}\bar{0}\bar{3}4\bar{2}3 \sim \bar{1}\bar{0}\bar{4}2\bar{3}4 \sim \bar{1}\bar{0}\bar{2}3\bar{4}2 \sim 013\bar{4}2\bar{3}$. Using these relations, we can obtain the elements in the set stabiliser, $N^{(012\bar{3}4\bar{2})}$. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:
$012\bar{3}4\bar{2} \sim \bar{1}\bar{0}\bar{3}4\bar{2}3 \Rightarrow (0\bar{1})\,(\bar{0}1)\,(2\bar{3}4\bar{2}3\bar{4})$
If we conjugate $Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2$ by this permutation, $(0\bar{1})\,(\bar{0}1)\,(2\bar{3}4\bar{2}3\bar{4})$, we obtain the following: $N\left(t_0t_1t_2\bar{t}_3t_4\bar{t}_2\right)^{(0\bar{1})(\bar{0}1)(2\bar{3}4\bar{2}3\bar{4})} = N\bar{t}_1\bar{t}_0\bar{t}_3t_4\bar{t}_2t_3$, but the relation tells us that $\bar{1}\bar{0}\bar{3}4\bar{2}3 \sim 012\bar{3}4\bar{2}$
$\Rightarrow N\bar{t}_1\bar{t}_0\bar{t}_3t_4\bar{t}_2t_3 = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2$
$\Rightarrow (0\bar{1})\,(\bar{0}1)\,(2\bar{3}4\bar{2}3\bar{4}) \in N^{(012\bar{3}4\bar{2})}$
So, $N^{(012\bar{3}4\bar{2})} \geq \left\langle (0\bar{1})\,(\bar{0}1)\,(2\bar{3}4\bar{2}3\bar{4}), N^{012\bar{3}4\bar{2}} \right\rangle \Rightarrow \left|N^{(012\bar{3}4\bar{2})}\right| \geq 6$.
Hence $\left|N^{(012\bar{3}4\bar{2})}\right| = 6$. To find the orbits of $N^{(012\bar{3}4\bar{2})}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(012\bar{3}4\bar{2})}$. The orbits of $N^{(012\bar{3}4\bar{2})}$ are $\{0, \bar{1}\}, \{\bar{0}, 1\}$, and $\{2, \bar{3}, 4, \bar{2}, 3, \bar{4}\}$. The number of single cosets in the double coset $[012\bar{3}4\bar{2}]$ is obtained by dividing $\frac{|N|}{\left|N^{(012\bar{3}4\bar{2})}\right|} = \frac{120}{6} = 20$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012\bar{3}4\bar{2}] = Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2N = \{N\,(t_0t_1t_2\bar{t}_3t_4\bar{t}_2)^n\,|n \in N\}$.

Next, we must take a representative of each orbit of $N^{(012\bar{3}4\bar{2})}$ and apply it to $Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2 \cdot t_0 = Nt_0t_1t_2\bar{t}_4t_0 \in [012\bar{4}0]$.
Since the orbit of 0 is of length two,
two t_i's will collapse the cayley graph from $[012\bar{3}4\bar{2}]$ to $[012\bar{4}0]$.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2 \cdot t_1 = Nt_0t_1t_2t_3t_4\bar{t}_0 \in [01234\bar{0}]$.
Since the orbit of 1 is of length two,
two t_i's will map the cayley graph from $[012\bar{3}4\bar{2}]$ to $[01234\bar{0}]$.

$Nt_0t_1t_2\bar{t}_3t_4\bar{t}_2 \cdot t_2 = Nt_0t_1t_2\bar{t}_3t_4 \in [012\bar{3}4]$.
Since the orbit of 2 is of length six,
six t_i's will collapse the cayley graph from $[012\bar{3}4\bar{2}]$ to $[012\bar{3}4]$.

Now we look at the double coset $Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0N$ which we denoted by $[0\bar{1}2\bar{3}4\bar{0}]$. To obtain the elements in $[0\bar{1}2\bar{3}4\bar{0}]$, we must first find the point stabiliser of $0, \bar{1}, 2, \bar{3}, 4$ and $\bar{0}$, $N^{0\bar{1}2\bar{3}4\bar{0}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, \bar{1}, 2, \bar{3}, 4$ and $\bar{0}$, hence $\left|N^{0\bar{1}2\bar{3}4\bar{0}}\right| = 1$. Using magma we obtain the relations $0\bar{1}2\bar{3}4\bar{0} \sim 2\bar{3}4\bar{0}1\bar{2} \sim 1\bar{2}3\bar{4}0\bar{1} \sim 4\bar{0}1\bar{2}3\bar{4} \sim 3\bar{4}0\bar{1}2\bar{3}$. Using these relations, we can obtain the elements in the set stabiliser, $N^{(0\bar{1}2\bar{3}4\bar{0})}$. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:

$0\bar{1}2\bar{3}4\bar{0} \sim 2\bar{3}4\bar{0}1\bar{2} \Rightarrow (02413)\,(\bar{0}\bar{2}\bar{4}\bar{1}\bar{3})$

$0\bar{1}2\bar{3}4\bar{0} \sim 1\bar{2}3\bar{4}0\bar{1} \Rightarrow (01234)\,(\bar{0}\bar{1}\bar{2}\bar{3}\bar{4})$

$0\bar{1}2\bar{3}4\bar{0} \sim 4\bar{0}1\bar{2}3\bar{4} \Rightarrow (04321)\,(\bar{0}\bar{4}\bar{3}\bar{2}\bar{1})$

$0\bar{1}2\bar{3}4\bar{0} \sim 3\bar{4}0\bar{1}2\bar{3} \Rightarrow (03142)\,(\bar{0}\bar{3}\bar{1}\bar{4}\bar{2})$

If we conjugate $Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0$ by the first permutation, $(02413)\,(\bar{0}\bar{2}\bar{4}\bar{1}\bar{3})$, we obtain the following: $N\,(t_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0)^{(02413)(\bar{0}\bar{2}\bar{4}\bar{1}\bar{3})} = Nt_2\bar{t}_3t_4\bar{t}_0t_1\bar{t}_2$, but the relation tells us that $2\bar{3}4\bar{0}1\bar{2} \sim 0\bar{1}2\bar{3}4\bar{0}$

$\Rightarrow Nt_2\bar{t}_3t_4\bar{t}_0t_1\bar{t}_2 = Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0$

$\Rightarrow (02413)\,(\bar{0}\bar{2}\bar{4}\bar{1}\bar{3}) \in N^{(0\bar{1}2\bar{3}4\bar{0})}$

So, $N^{(0\bar{1}\bar{2}\bar{3}4\bar{0})} \geq \left\langle (02413)\,(\bar{0}\bar{2}\bar{4}\bar{1}\bar{3}), N^{0\bar{1}\bar{2}\bar{3}4\bar{0}} \right\rangle \Rightarrow \left|N^{(0\bar{1}\bar{2}\bar{3}4\bar{0})}\right| \geq 5.$
Hence $\left|N^{(0\bar{1}\bar{2}\bar{3}4\bar{0})}\right| = 5$. To find the orbits of $N^{(0\bar{1}\bar{2}\bar{3}4\bar{0})}$ we conjugate $0, 1, 2, 3$ and 4 by $N^{(0\bar{1}\bar{2}\bar{3}4\bar{0})}$. The orbits of $N^{(0\bar{1}\bar{2}\bar{3}4\bar{0})}$ are $\{0, 1, 2, 3, 4\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. The number of single cosets in the double coset $[0\bar{1}\bar{2}\bar{3}4\bar{0}]$ is obtained by dividing $\frac{|N|}{\left|N^{(0\bar{1}\bar{2}\bar{3}4\bar{0})}\right|} = \frac{120}{5} = 24$.
To obtain the single cosets in this double coset we apply the permutations in S_5:
$[0\bar{1}\bar{2}\bar{3}4\bar{0}] = Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0N = \{N\,(t_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0)^n \,|n \in N\}$

Next, we must take a representative of each orbit of $N^{(0\bar{1}\bar{2}\bar{3}4\bar{0})}$ and apply it to $Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0$ to determine if it will extend or collapse the Cayley graph:

$Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0 \cdot t_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4 \in [0\bar{1}2\bar{3}4]$.
Since the orbit of 0 is of length five,
five t_i's will collapse the cayley graph from $[0\bar{1}\bar{2}\bar{3}4\bar{0}]$ to $[0\bar{1}\bar{2}\bar{3}4]$.

$Nt_0\bar{t}_1t_2\bar{t}_3t_4\bar{t}_0 \cdot \bar{t}_0 = Nt_0\bar{t}_1t_2\bar{t}_3t_4t_0 = Nt_0\bar{t}_1t_2\bar{t}_4t_3 \in [0\bar{1}2\bar{4}3]$.
Since the orbit of $\bar{0}$ is of length five,
five t_i's will collapse the cayley graph from $[0\bar{1}\bar{2}\bar{3}4\bar{0}]$ to $[0\bar{1}\bar{2}\bar{4}3]$.

Now we look at the double coset $Nt_0t_1t_2t_3t_0t_1t_4N$ which we denoted by [0123014]. To obtain the elements in [0123014], we must first find the point stabiliser of $0, 1, 2, 3$ and 4, $N^{0123014}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0, 1, 2, 3$ and 4, hence $\left|N^{0123014}\right| = 1$. Using magma we obtain the following relations:
$0123014 \sim 3210324 \sim 1032104 \sim 2301234$.
Using these relations, we can obtain the elements in the set stabiliser. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:
$0123014 \sim 3210324 \Rightarrow (03)\,(12)\,(\bar{0}\bar{3})\,(\bar{1}\bar{2})$
$0123014 \sim 1032104 \Rightarrow (01)\,(23)\,(\bar{0}\bar{1})\,(\bar{2}\bar{3})$
$0123014 \sim 2301234 \Rightarrow (02)\,(13)\,(\bar{0}\bar{2})\,(\bar{1}\bar{3})$
If we conjugate $Nt_0t_1t_2t_3t_0t_1t_4$ by the first permutation, $(03)\,(12)\,(\bar{0}\bar{3})\,(\bar{1}\bar{2})$, we obtain the following: $N\,(t_0t_1t_2t_3t_0t_1t_4)^{(03)(12)(\bar{0}\bar{3})(\bar{1}\bar{2})} = Nt_3t_2t_1t_0t_3t_2t_4$, but the relation tells us that $3210324 \sim 0123014$
$\Rightarrow Nt_3t_2t_1t_0t_3t_2t_4 = Nt_0t_1t_2t_3t_0t_1t_4$
$\Rightarrow (03)\,(12)\,(\bar{0}\bar{3})\,(\bar{1}\bar{2}) \in N^{(0123014)}$

So, $N^{(0123014)} \geq \langle (03)(12)(\bar{0}\bar{3})(\bar{1}\bar{2}), N^{0123014} \rangle \Rightarrow |N^{(0123014)}| \geq 2$.
If we conjugate $Nt_0t_1t_2t_3t_0t_1t_4$ by the second permutation, $(01)(23)(\bar{0}\bar{1})(\bar{2}\bar{3})$, we obtain the following: $N(t_0t_1t_2t_3t_0t_1t_4)^{(01)(23)(\bar{0}\bar{1})(\bar{2}\bar{3})} = Nt_1t_0t_3t_2t_1t_0t_4$, but the relation tells us that $1032104 \sim 0123014$
$\Rightarrow Nt_1t_0t_3t_2t_1t_0t_4 = Nt_0t_1t_2t_3t_0t_1t_4$
$\Rightarrow (01)(23)(\bar{0}\bar{1})(\bar{2}\bar{3}) \in N^{(0123014)}$
So, $N^{(0123014)} = \langle (03)(12)(\bar{0}\bar{3})(\bar{1}\bar{2}), (01)(23)(\bar{0}\bar{1})(\bar{2}\bar{3}), N^{0123014} \rangle \Rightarrow |N^{(0123014)}| = 4$.
To find the orbits of $N^{(0123014)}$ we conjugate $0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}$, and $\bar{4}$ by $N^{(0123014)}$. The orbits of $N^{(0123014)}$ are $\{0,1,2,3\}, \{\bar{0},\bar{1},\bar{2},\bar{3}\}, \{4\}$, and $\{\bar{4}\}$. The number of single cosets in the double coset [0123014] is obtained by dividing $\frac{|N|}{|N^{(0123014)}|} = \frac{120}{4} = 30$. To obtain the single cosets in this double coset we apply the permutations in S_5:
$[0123014] = Nt_0t_1t_2t_3t_0t_1t_4N = \{N(t_0t_1t_2t_3t_0t_1t_4)^n | n \in N\}$

Next, we must take a representative of each orbit of $N^{(0123014)}$ and apply it to $Nt_0t_1t_2t_3t_0t_1t_4$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3t_0t_1t_4 \cdot t_0 = Nt_0t_1t_2t_3t_0t_1\bar{t}_4 \in [012304]$.
Since the orbit of 0 is of length four,
four t_i's will collapse the cayley graph from [0123014] to [012304].

$Nt_0t_1t_2t_3t_0t_1t_4 \cdot t_4 = Nt_0t_1t_2t_3t_0t_1\bar{t}_4 \in [0123014]$.
Since the orbit of 4 is of length one,
one t_i will map the cayley graph from [0123014] to itself, [0123014].

$Nt_0t_1t_2t_3t_0t_1t_4 \cdot \bar{t}_0 = Nt_0t_1t_2t_3t_0t_4 \in [012304]$.
Since the orbit of $\bar{0}$ is of length four,
four t_i's will collapse the cayley graph from [0123014] to [012304].

$Nt_0t_1t_2t_3t_0t_1t_4 \cdot \bar{t}_4 = Nt_0t_1t_2t_3t_0t_1 \in [012301]$.
Since the orbit of $\bar{4}$ is of length one,
one t_i will collapse the cayley graph from [0123014] to [012301].

Now we look at the double coset $Nt_0t_1t_2t_3t_0t_4\bar{t}_1N$ which we denoted by $[012304\bar{1}]$.

To obtain the elements in $[012304\bar{1}]$, we must first find the point stabiliser of $0,1,2,3,4$ and $\bar{1}$, $N^{012304\bar{1}}$. The point stabiliser consists of permutations in $N = S_5$, which fix $0,1,2,3,4$ and $\bar{1}$, hence $\left|N^{012304\bar{1}}\right| = 1$. Using magma we obtain the relations $012304\bar{1} \sim \bar{2}\bar{4}0\bar{1}\bar{2}\bar{3}4 \sim \bar{2}\bar{3}0\bar{4}\bar{2}\bar{1}3 \sim \bar{2}\bar{1}0\bar{3}\bar{2}\bar{4}1 \sim 0421034 \sim 0324013$. Using these relations, we can obtain the elements in the set stabiliser, $N^{(012304\bar{1})}$. These relations can be represented as permutations in S_5. These permutaions are obtained as follows:

$012304\bar{1} \sim \bar{2}\bar{4}0\bar{1}\bar{2}\bar{3}4 \Rightarrow (0\bar{2})\,(\bar{0}2)\,(1\bar{4}3\bar{1}4\bar{3})$

$012304\bar{1} \sim \bar{2}\bar{3}0\bar{4}\bar{2}\bar{1}3 \Rightarrow (0\bar{2})\,(\bar{0}2)\,(1\bar{3}4\bar{1}3\bar{4})$

$012304\bar{1} \sim \bar{2}\bar{1}0\bar{3}\bar{2}\bar{4}1 \Rightarrow (0\bar{2})\,(\bar{0}2)\,(1\bar{1})\,(3\bar{3})\,(4\bar{4})$

$012304\bar{1} \sim 0421034 \Rightarrow (143)\,(\bar{1}4\bar{3})$

$012304\bar{1} \sim 0324013 \Rightarrow (134)\,(\bar{1}3\bar{4})$

If we conjugate $Nt_0t_1t_2t_3t_0t_4\bar{t}_1$ by the first permutation, $(0\bar{2})\,(\bar{0}2)\,(1\bar{4}3\bar{1}4\bar{3})$, we obtain the following: $N\left(t_0t_1t_2t_3t_0t_4\bar{t}_1\right)^{(0\bar{2})(\bar{0}2)(1\bar{4}3\bar{1}4\bar{3})} = N\bar{t}_2\bar{t}_4\bar{t}_0\bar{t}_1\bar{t}_2\bar{t}_3t_4$, but the relation tells us that $\bar{2}\bar{4}0\bar{1}\bar{2}\bar{3}4 \sim 012304\bar{1}$

$\Rightarrow N\bar{t}_2\bar{t}_4\bar{t}_0\bar{t}_1\bar{t}_2\bar{t}_3t_4 = Nt_0t_1t_2t_3t_0t_4\bar{t}_1$

$\Rightarrow (0\bar{2})\,(\bar{0}2)\,(1\bar{4}3\bar{1}4\bar{3}) \in N^{(0123014)}$

So, $N^{(012304\bar{1})} \geq \left\langle (0\bar{2})\,(\bar{0}2)\,(1\bar{4}3\bar{1}4\bar{3}), N^{012304\bar{1}} \right\rangle \Rightarrow \left|N^{(012304\bar{1})}\right| \geq 6.$

Therefore $\left|N^{(012304\bar{1})}\right| \geq 6$

To find the orbits of $N^{(012304\bar{1})}$ we conjugate $0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3}$, and $\bar{4}$ by $N^{(012304\bar{1})}$. The orbits of $N^{(012304\bar{1})}$ are $\{0\bar{2}\}, \{\bar{0}2\}$, and $\{1\bar{4}3\bar{1}4\bar{3}\}$.The number of single cosets in the double coset $[012304\bar{1}]$ is obtained by dividing $\frac{|N|}{\left|N^{(012304\bar{1})}\right|} = \frac{120}{6} = 20$. To obtain the single cosets in this double coset we apply the permutations in S_5:

$[012304\bar{1}] = Nt_0t_1t_2t_3t_0t_4\bar{t}_1N = \{N\left(t_0t_1t_2t_3t_0t_4\bar{t}_1\right)^n | n \in N\}$

Next, we must take a representative of each orbit of $N^{(012304\bar{1})}$ and apply it to $Nt_0t_1t_2t_3t_0t_4\bar{t}_1$ to determine if it will extend or collapse the Cayley graph.

$Nt_0t_1t_2t_3t_0t_4\bar{t}_1 \cdot t_0 = Nt_0t_1t_2t_3t_4\bar{t}_0 \in [01234\bar{0}]$.

Since the orbit of 0 is of length two,

two t_i's will collapse the cayley graph from $[012304\bar{1}]$ to $[01234\bar{0}]$.

$Nt_0t_1t_2t_3t_0t_4\bar{t}_1 \cdot t_1 = Nt_0t_1t_2t_3t_0t_4 \in [012304]$.

Since the orbit of 1 is of length six,

six t_i's will collapse the cayley graph from $[012304\bar{1}]$ to $[012304]$.

$Nt_0t_1t_2t_3t_0t_4\bar{t}_1 \cdot \bar{t}_0 = Nt_0t_1t_2t_3\bar{t}_4t_1 \in [0123\bar{4}1]$.
Since the orbit of $\bar{0}$ is of length two,
two t_i's will collapse the cayley graph from $[012304\bar{1}]$ to $[0123\bar{4}1]$.

Since the set of right cosets is closed under right coset multiplication, we have completed the double coset enumeration.The completed the Cayley Graph, is shown on the following page. The index of $N = S_5$ in G is $2,520$, which is obtained by ading all the single cosets in each of the double cosets. Recall that the order of N is $|N| = 5! = 120$. Thus $|G| \leq 2520 \times |N| = 2520 \times 120 = 302400$. Therefore, $|G| \leq 302400$. Using the same procedure as before, we can show that $|G| \geq 302400$. Hence, we can conclude that the order of G is equal to 302400, $|G| = 302400$.

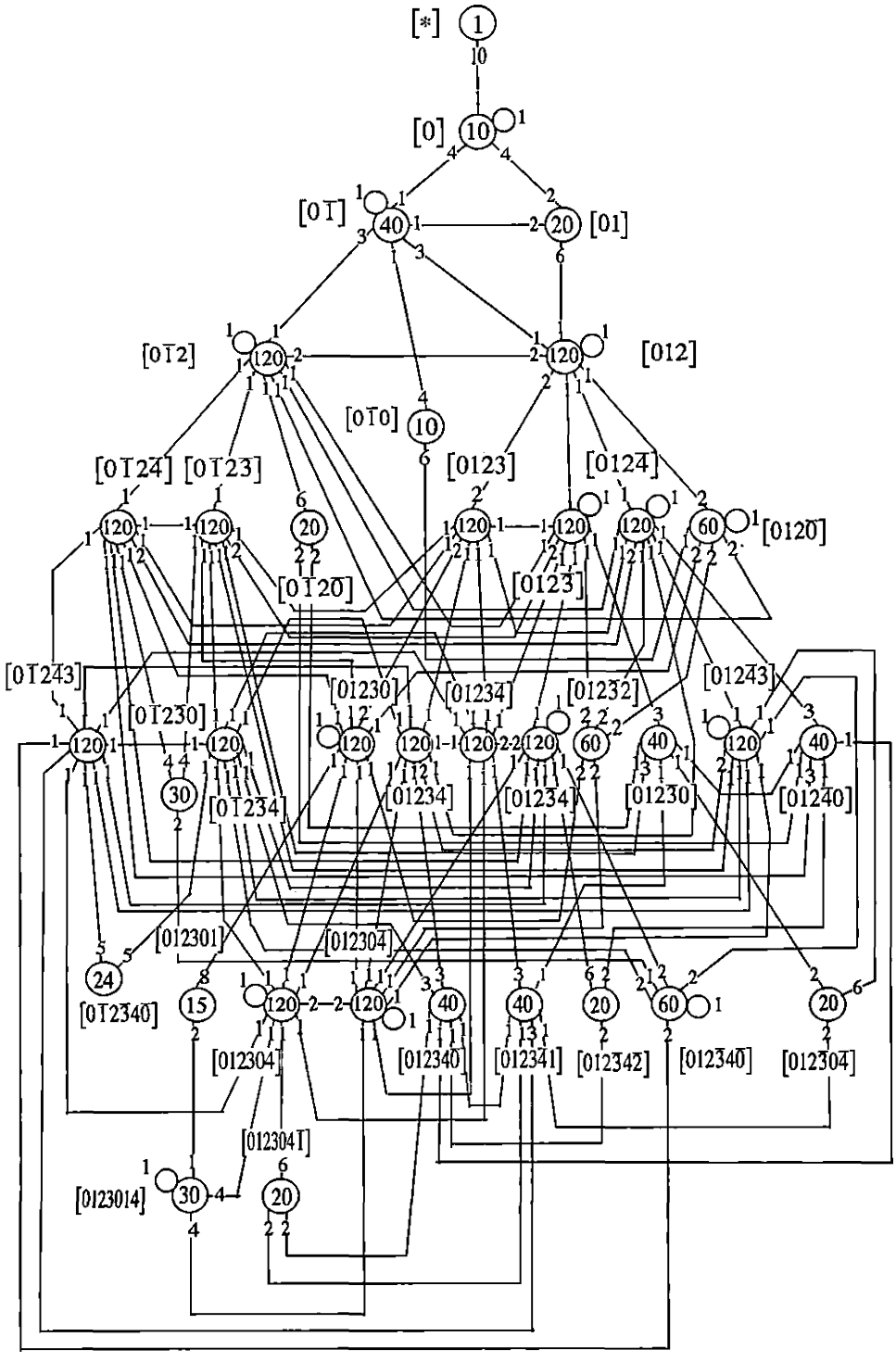

Figure 5.1: The Cayley Graph of $A_7 \times S_5$ over S_5

Chapter 6

The Construction of A_7

We want to that the symmetric representationof A_7 is given by $\frac{A_7 \times S_5}{(2,3,4)(\bar{2}\bar{3}\bar{4})\bar{t}_4 t_3 \bar{t}_2 t_4;(0,0)(1,\bar{1})(2,\bar{2})(3,\bar{4})(\bar{4},\bar{3})t_3 t_4 t_3;(0,1,2)(\bar{0},\bar{1},\bar{2})t_1 t_0 t_2 t_1}$, where $X \sim (01234)\,(\bar{0}\bar{1}\bar{2}\bar{3}\bar{4})$ and $Y \sim (0\bar{0})\,(1\bar{1})\,(2\bar{2})\,(3\bar{4})\,(4\bar{3})$.

The symmetric presentation of the progenitor $3^{*5} :_m S_5$ is given by

$3^{*5} :_m S_5 \cong \left\langle x, y, t | x^5 = y^2 = (yx)^4 = [x,y]^3 = 1 = t^3 = (yt)^2 = (yt^x)^2, \left(yt^{x^2}\right)^2 \right\rangle$.

The control subgroup N is S_5 on ten letters, t_i and $\bar{t}_i$ for $i = 0, .., 4$. We must perform a double coset enumeration of G over S_5. The double coset enumeration consists of finding all double cosets $[w]$ and figuring out how many single cosets each of these contains. Notice that the double coset enumeration is complete when the set of right cosets is closed under right multiplication. That is, when none of the t_i's can expand the cayley graph any further. Notice that from the definition of 3^{*5}, we obtain the property: $t_i^3 = 1$ for $i = 0, ..., 4$. Then multiplying by the inverse of the t_i's, results in a new property: $t_i^3 \cdot t_i^{-1} = 1 \cdot t_i^{-1} \Longrightarrow t_i^2 = t_i^{-1}$. Thus $t_i^2 = t_i^{-1}$ and we will denote the inverses t_i^{-1} as $\bar{t}_i$. The control subgroup is the permutation group S_5 on ten letters.

We will proceed to expanding our relators and obtining our relations.

The first relation is as follows $(2,3,4)\,(\bar{2},\bar{3},\bar{4})\,\bar{t}_4 t_3 \bar{t}_2 t_4 = e$.

If we use the operation of right multiplication we can further simplify this relation:

$\Longrightarrow (2,3,4)\,(\bar{2},\bar{3},\bar{4})\,\bar{t}_4 t_3 \bar{t}_2 t_4 \cdot \bar{t}_4 = e \cdot \bar{t}_4$

$\Longrightarrow (2,3,4)\,(\bar{2},\bar{3},\bar{4})\,\bar{t}_4 t_3 \bar{t}_2 = \bar{t}_4$

$\Longrightarrow (2,3,4)\,(\bar{2},\bar{3},\bar{4})\,\bar{t}_4 t_3 \bar{t}_2 \cdot t_2 = \bar{t}_4 \cdot t_2$

$\Longrightarrow (2,3,4)\,(\bar{2},\bar{3},\bar{4})\,\bar{t}_4 t_3 = \bar{t}_4 t_2$

Thus, we obtain the following relation $(2,3,4)\,(\bar{2},\bar{3},\bar{4})\,\bar{t}_4 t_3 = \bar{t}_4 t_2$.

The second relation is as follows $(5,\bar{5})\,(1,\bar{1})\,(2,\bar{2})\,(3,\bar{4})\,(4,\bar{3})\,t_3\bar{t}_4 t_3 = e$.

If we use the operation of right multiplication we can further simplify this relation:

$\Longrightarrow (5,\bar{5})\,(1,\bar{1})\,(2,\bar{2})\,(3,\bar{4})\,(4,\bar{3})\,t_3\bar{t}_4 t_3 \cdot \bar{t}_3 = e \cdot \bar{t}_3$

$\Longrightarrow (5,\bar{5})\,(1,\bar{1})\,(2,\bar{2})\,(3,\bar{4})\,(4,\bar{3})\,t_3\bar{t}_4 = \bar{t}_3$

$\Longrightarrow (5,\bar{5})\,(1,\bar{1})\,(2,\bar{2})\,(3,\bar{4})\,(4,\bar{3})\,t_3\bar{t}_4 \cdot t_4 = \bar{t}_3 \cdot t_4$

$\Longrightarrow (5,\bar{5})\,(1,\bar{1})\,(2,\bar{2})\,(3,\bar{4})\,(4,\bar{3})\,t_3 = \bar{t}_3 t_4$

Thus, we obtain the following relation $(5,\bar{5})\,(1,\bar{1})\,(2,\bar{2})\,(3,\bar{4})\,(4,\bar{3})\,t_3 = \bar{t}_3 t_4$.

The third relation is as follows $(5,1,2)\,(\bar{5},\bar{1},\bar{2})\,\bar{t}_1 t_5 \bar{t}_2 t_1 = e$.

If we use the operation of right multiplication we can further simplify this relation:

$\Longrightarrow (5,1,2)\,(\bar{5},\bar{1},\bar{2})\,\bar{t}_1 t_5 \bar{t}_2 t_1 \cdot \bar{t}_1 = e \cdot \bar{t}_1$

$\Longrightarrow (5,1,2)\,(\bar{5},\bar{1},\bar{2})\,\bar{t}_1 t_5 \bar{t}_2 = \bar{t}_1$

$\Longrightarrow (5,1,2)\,(\bar{5},\bar{1},\bar{2})\,\bar{t}_1 t_5 \bar{t}_2 \cdot t_2 = \bar{t}_1 \cdot t_2$

$\Longrightarrow (5,1,2)\,(\bar{5},\bar{1},\bar{2})\,\bar{t}_1 t_5 = \bar{t}_1 t_2$

Thus, we obtain the following relation $(5,1,2)\,(\bar{5},\bar{1},\bar{2})\,\bar{t}_1 t_5 = \bar{t}_1 t_2$.

The fourth relation is as follows $(5,\bar{4},2,\bar{5},4,\bar{2})\,(1,\bar{3})\,(3,\bar{1})\,\bar{t}_5 t_4 \bar{t}_2 t_5 = e$.

If we use the operation of right multiplication we can further simplify this relation:

$\Longrightarrow (5,\bar{4},2,\bar{5},4,\bar{2})\,(1,\bar{3})\,(3,\bar{1})\,\bar{t}_5 t_4 \bar{t}_2 t_5 \cdot \bar{t}_5 = e \cdot \bar{t}_5$

$\Longrightarrow (5,\bar{4},2,\bar{5},4,\bar{2})\,(1,\bar{3})\,(3,\bar{1})\,\bar{t}_5 t_4 \bar{t}_2 = \bar{t}_5$

$\Longrightarrow (5,\bar{4},2,\bar{5},4,\bar{2})\,(1,\bar{3})\,(3,\bar{1})\,\bar{t}_5 t_4 \bar{t}_2 \cdot t_2 = \bar{t}_5 \cdot t_2$

$\Longrightarrow (5,\bar{4},2,\bar{5},4,\bar{2})\,(1,\bar{3})\,(3,\bar{1})\,\bar{t}_5 t_4 = \bar{t}_5 t_2$

Thus, we obtain the following relation $(5,\bar{4},2,\bar{5},4,\bar{2})\,(1,\bar{3})\,(3,\bar{1})\,\bar{t}_5 t_4 = \bar{t}_5 t_2$.

6.1 The Double Coset Enumeration of G over S_5

First, note that $N = \langle (0,\bar{0})\,(1,\bar{1})\,(2,\bar{2})\,(3,\bar{4})\,(4,\bar{3}), (0,1,2,3,4)\,(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})\rangle$. The empty word is $NeN = \{Nen \mid n \in N\} = \{Nn \mid n \in N\} = \{N\}$. We will denote this double coset by $[*]$. The number of elements in $[*]$ is $\frac{|N|}{|N|} = \frac{120}{120} = 1$, hence $[*]$ consists of the single coset, N. Since $N \cong S_5$,it acts transitively on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Therefore, N has a single orbit, namely $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ in $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$.

Take a representative from this orbit and apply it to N, and determine to which double coset it belongs to (in other words, if the Cayley graph is expanded):

$N \cdot t_0 = Nt_0 \in [0]$.
Since the orbit of 0 is of length ten,
ten t_i's will extend the Cayley graph from $[*]$ to $[0]$.

Next, look at the double coset Nt_0N , the words of length one, which we will denote by [0]. To obtain the elements in [0], we must first find the point stabiliser of 0, N^0. The point stabiliser, N^0, consists of permutations in $N = S_5$ which fix 0 and permute $1, 2, 3, 4$. This means that $\left|N^0\right| = 4! = 24$. In this case, the point stabiliser is equal to the set stabiliser, $N^{(0)}$. $N^{(0)} = N^0 = \langle (1,2,3,4), (\bar{1},\bar{2},\bar{3},\bar{4})\rangle = \langle (1,2,3)(1,4), (\bar{1},\bar{2},\bar{3})(\bar{1},\bar{4})\rangle$ $= \langle (1,2)(1,3)(1,4), (\bar{1},\bar{2})(\bar{1},\bar{3})(\bar{1},\bar{4})\rangle \cong A_4$ on 8 letters. Thus, $\left|N^{(0)}\right| = \frac{4!}{2} = 12$. Hence To find the orbits of $N^{(0)}$, we conjugate $0, 1, 2, 3$ and 4 by $N^{(0)}$. The orbits of $N^{(0)}$ are $\{0\}, \{\bar{0}\}, \{1,2,3,4\}$ and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$. The number of single cosets in the double coset [0] is obtained by dividing $\frac{|N|}{|N^{(0)}|} = \frac{120}{12} = 10$. To obtain the elements in this double coset, we must first find the right cosets, also known as transversals, of $N^{(0)}$ in N. Therefore $[0] = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}, N\bar{t_3}, N\bar{t_4}\}$.

Now we must take a representative from each orbit of $N^{(0)}$ and apply it to Nt_0 to determine if it will extend or collapse the Cayley graph.

$Nt_0 \cdot t_0 = N\bar{t_0} \in [0]$.
Since the orbit of 0 is of length 1,
one t_i will map [0] to [0].

$Nt_0 \cdot \bar{t_0} = N \in [*]$
Since the orbit of $\bar{0}$ is of length 1,
one t_i will collapse the Cayley graph from [0] to $[*]$.

$Nt_0 \cdot t_1 = Nt_0t_1$
This is a new double coset which we will label as [01].
Since the orbit of 1 is of length 4,

four t_i's will extend the Cayley graph from [0] to [01].

$Nt_0 \cdot \bar{t}_1 = Nt_0\bar{t}_1 = Nt_0 \in [0]$
Since the orbit of $\bar{1}$ is of length 4,
four t_i's will map [0] to [0].

Now, we look at the double coset Nt_0t_1N which we denoted [01]. To obtain the elements in [01], we must first find the point stabiliser of 0 and 1, N^{01}. The point stabiliser consists of permutations in $N = S_5$ which fix 0 and 1 and permute $2, 3, 4$. This means that $\left|N^{01}\right| = 3! = 6$. The set stabiliser, $N^{(01)} \geq N^{01}$. Due to the relation mentioned above, the set stabiliser increases:

$$N\left(t_0t_1\right)^{(01)} = Nt_1t_0 = Nt_0t_1 \Longrightarrow (01) \in N^{(01)} \Longrightarrow N^{(01)} \geq \left\langle N^{01}, (01)\right\rangle.$$

Therefore, $\left|N^{(01)}\right| = 6 \cdot 2 = 12$. To find the orbits of $N^{(01)}$, we conjugate $0, 1, 2$ and 3 by $N^{(01)}$. The orbits of $N^{(01)}$ are $\{0, 1, \bar{1}, \bar{0}\}$ and $\{2, 3, 4, \bar{2}, \bar{3}, \bar{4}\}$. The number of single cosets in the double coset [01] is obtained by dividing $\frac{|N|}{\left|N^{(01)}\right|} = \frac{120}{12} = 10$. We used magma to find the transversals of $N^{(01)}$ in N, the transversals are:
$Id(N), (0,1,2,3,4)\,(\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}), (1,3,0,2,4)$
$(\bar{1},\bar{3},\bar{0},\bar{2},\bar{4}), (1,4,2,0,3)\,(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3}),$
$(1,\bar{4})\,(2,\bar{3},0,\bar{2},3,\bar{0})\,(4,\bar{1}), (1,0,4,3,2)\,(\bar{1},\bar{0},\bar{4},\bar{3},\bar{2}),$
$(1,\bar{0},3,\bar{2})\,(2,\bar{1},0,\bar{3})\,(4,\bar{4}), (1,\bar{1})\,(2,\bar{0},4,\bar{3})\,(3,\bar{2},0,\bar{4}), (1,\bar{2},3,\bar{4})\,(2,\bar{3},4,\bar{1})\,(0,\bar{0})$, and
$(1,\bar{3},4,\bar{0})\,(2,\bar{2})\,(3,\bar{4},0,\bar{1})$.
Now, we apply the transversals to Nt_0t_1 by conjugation to obtain the single cosets in [01]:
$[01] = Nt_0t_1N = \{N\left(t_0t_1\right)^n | n \in N\}$
$\Longrightarrow [01] = \{Nt_0t_1, Nt_1t_2, Nt_2t_3, Nt_3t_4, N\bar{t}_2\bar{t}_4, Nt_4t_0, N\bar{t}_3\bar{t}_0, N\bar{t}_4\bar{t}_1, N\bar{t}_0\bar{t}_2, N\bar{t}_1\bar{t}_3\}$

Now, we must take a representative of each orbit of $N^{(01)}$ and apply it to Nt_0t_1 to determine if it will extend or collapse the Cayley graph:

$Nt_0t_1 \cdot \bar{t}_1 = Nt_0\left(t_1 \cdot \bar{t}_1\right) = Nt_0 \in [0]$.
Since the orbit of $\bar{1}$ is of length 4,
four t_i's collapse the Cayley graph from [01] to [0].

$Nt_0t_1 \cdot t_2 = Nt_0t_1t_2 \in [01]$
Since the orbit of 2 is of length 6,
six t_i's will map [01] to [01].

Thus, we obtain the following Cayley graph:

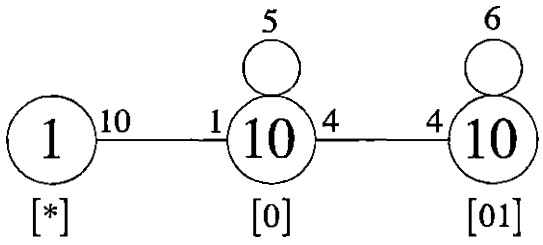

Figure 6.1: The Cayley Graph of A_7 over S_5

So the index of $N = S_5$ in G is

$$\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} = 1 + 10 + 10 = 21$$

Recall that the order of N is $|N| = 5! = 120$. The Cayley diagram gives the maximum order of G, thus $|G| \leq 21 \times 120 = 2520$. In other wors, it establishes that $|G| \leq 2520$. Using the same process as shown in the previous chapters, we can show that $|G| \geq 2520$. Hence, we can conclude that that $|G| = 2520$.

Chapter 7

The Construction of $3 \times M_{22}$

I began my construction of the group $3 \times M_{22}$, but I was unable to finish it. I have included the magma code for the incomplete construction in the appendix. Please see Appendix B.

Appendix A: MAGMA Code for $(2 \times S_6) : 2^4$

```
S:=Sym(6);
xx:=S!(6,1,2,3,4,5);
yy:=S!(6,1);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^6,y^2,(x*y)^5,(y*y^x)^3,(y*y^(x^2))^2,
t^2,(x*y,t),(t,(y^(x^4))),(x*t)^6,(t*t^x)^2>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[f(t^(x^i)): i in [1,2,3,4,5,6]];
prodim := function(pt, Q, I);

/* Return the image of pt under permutations
Q[I] applied sequentially.  */
v := pt;
for i in I do  v := v^(Q[i]);
end for;
return v;
end function;

#G;

/* ====================================
       Code for Coset Stabilisers
====================================*/
N6:=Stabiliser(N,6);N6;
N6c:=sub<N|N6>;

N61:=Stabiliser(N6,1);N61;
N61c:=sub<N|N61>;
```

```
N612:=Stabiliser(N61,2);N612;#N612;
N612c:=sub<N|N612>;

/* ==================================
          Code for Transversals
==================================*/
T6s := Transversal(N,N6); T6s;

T61:=Transversal(N,N61c);T61;

T612:=Transversal(N,N612c);T612;

/* ==========================================
  Code for finding the number of single cosets
==========================================*/

cst := [null : i in [1 .. 32]] where null is [Integers() | ];
for i := 1 to 6 do cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
for i in [1..32] do if cst[i] ne [] then m:=m+1; end if; end for; m;

for i in [1..#T6s] do
ss:=[6]^T6s[i];
cst[prodim(1,ts, ss)]:=ss;
end for;
m:=0;
for i in [1..32] do if cst[i] ne [] then m:=m+1; end if; end for; m;

for i in [1..#T61] do
ss:=[6,1]^T61[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..32] do if cst[i] ne [] then m:=m+1; end if; end for; m;

for i in [1..#T612] do
ss:=[6,1,2]^T612[i];
cst[prodim(1, ts, ss)] :=ss;
end for;
m:=0;
for i in [1..32] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* ========================================
```

```
Permutation Representation to Symmetric Representation
==========================================*/
per2sym := function(G1,N, p);

ww := cst[1^p];
tt := p * &*[G1|(ts[ww[#ww - l + 1]])^-1: l in [1 .. #ww]];
zz := N![rep{j: j in [1..6] | (1^ts[i])^tt eq 1^ts[j]}: i in [1..6]];
return <zz, ww>;
end function;

/* ==========================================
Symmetric Representation to Permutation Representation
==========================================*/
sym2per := function(G1,N, yyy);

xxx := N ! yyy[1]; uuu := yyy[2];
p := [1 : i in [1 .. 32]];
ts:=[f(t^(x^i)): i in [1,2,3,4,5,6]];
for i := 1 to 6 do
p[prodim(1, ts, [i])] := prodim(1, ts, [i]^xxx);
end for;

T6s := Transversal(N,N6);
for i in [1..#T6s] do
ss:=[6]^T6s[i];
p[prodim(1,ts, ss)]:=prodim(1,ts, ss^xxx);
end for;

T61 := Transversal(N,N61c);
for i in [1..#T61] do
ss:=[6,1]^T61[i];
p[prodim(1, ts, ss)]:= prodim(1, ts, ss^xxx);
end for;

T612:=Transversal(N,N612c);
for i in [1..#T612] do
ss:=[6,1,2]^T612[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

return (G1 ! p) * &*[G1|ts[uuu[j]]: j in [1 .. #uuu]];
end function;
```

Appendix B: MAGMA Code for $3 \times M_{22}$

```
S:=Sym(28);
xx:=S!(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21)
(22,23,24,25,26,27,28);
yy:=S!(1,8)(2,13)(3,10)(4,5)(6,9)(11,12)(15,22)(16,27)(17,24)(18,19)
(20,23)(25,26);
N:=sub<S|xx,yy>; #N;

for g in N do g; end for;

NN<x,y>:=Group<x,y|x^7,y^2,(x*y)^3,(x,y)^4>; #NN;

G<x,y,t>:=Group<x,y,t|x^7,y^2,(x*y)^3,(x,y)^4,t^3,(t^(x^4),x*y),(t,y),
(x*t)^5,(x^-1*t)^5>;

V:=CosetSpace(G,sub<G|x,y>:CosetLimit:=10000000);

/* The following computes the order of G*/
G1:=CosetImage(V); #G1;

fx:=G1.1;
fy:=G1.2;
ft:=G1.3;
IN:=sub<G1|fx,fy>; #IN;

ts:=[Id(G1):i in [1..28]];
ts[7]:=ft;
ts[1]:=(ft^fx);
ts[2]:=((ft)^((fx)^2));
ts[3]:=((ft)^((fx)^3));
ts[4]:=((ft)^((fx)^4));
```

```
ts[5]:=((ft)^((fx)^5));
ts[6]:=((ft)^((fx)^6));
ts[8]:=ts[1]^fy;
ts[9]:=ts[6]^fy;
ts[10]:=ts[3]^fy;
ts[11]:=ts[10]^fx;
ts[12]:=ts[11]^fx;
ts[13]:=ts[12]^fx;
ts[14]:=ts[13]^fx;
ts[15]:=ts[1]^-1;
ts[16]:=ts[2]^-1;
ts[17]:=ts[3]^-1;
ts[18]:=ts[4]^-1;
ts[19]:=ts[5]^-1;
ts[20]:=ts[6]^-1;
ts[21]:=ts[7]^-1;
ts[22]:=ts[8]^-1;
ts[23]:=ts[9]^-1;
ts[24]:=ts[10]^-1;
ts[25]:=ts[11]^-1;
ts[26]:=ts[12]^-1;
ts[27]:=ts[13]^-1;
ts[28]:=ts[14]^-1;

cst := [null : i in [1 .. 5280]] where null is [Integers() | ];
prodim := function(pt, Q, I)

/* Return the image of pt under permutations Q[I]
applied sequentially.  */
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;

#G1;

/*  ================================
1.                          [*]
================================  */
Orbits(N);

/*  ================================
```

```
 2.                          [7]
================================  */
/*Finds the relations.*/
S:={[7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[7] eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if;
end for;
end for;

/* Coset stabiliser of [7]*/
N7:=Stabiliser(N,7); #N7; N7;

N7c:=sub<N|N7>; #N7c; N7c;

/* Obtains the transversals of N(7) */
T7s:= Transversal(N,N7); T7s;

/* number of cosets through [7], excluding[*]. */
for i in [1..#T7s] do
ss:=[7]^T7s[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

/*the following will yield the relations necessary to
translate the permutations given for m and n.*/
for i in [1 .. 5280] do if cst[i] ne [] then i, cst[i];
end if; end for;

/*orbits*/
Orbits(N7);

/*  ================================
 3.                      [21]
================================  */
S:={[21]};
SS:=S^N;
SSS:=Setseq(SS);
```

```
for i in [1..#SS] do
for g in IN do if ts[21] eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if;
end for;
end for;

/* Coset stabiliser of [21]*/
N21:=Stabiliser(N,21); #N21; N21;

N21c:=sub<N|N21>; #N21c; N21c;

for g in N21c do g; end for;

/* Obtains the transversals of N(21) */
T21s:= Transversal(N,N21c); #T21s; T21s;

/* number of cosets through [21], excluding[*]. */
for i in [1..#T21s] do
ss:=[21]^T21s[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/*orbits*/
Orbits(N21c);

/*================
4.  [7,1]
==============*/
S:={[7,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [56]. */
N71:=Stabiliser(N7,1);#N71;N71;
N71c:=sub<N|N71>;#N71c;N71c;
```

```
/* Obtains the transversals of N(71).*/
T71:=Transversal(N,N71c);#T71;T71;

/*Obtains the total number of cosets(through [71] & excluding [*])
by conjugation. */
for i in [1..#T71] do
ss:=[7,1]^T71[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

for g in N71c do g; end for;

/*Checks the orbits of N71c */
Orbits(N71c);

/*================
5.  [715]
==============*/
S:={[7,15]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[15] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [715]. */
N715:=Stabiliser(N7,15);#N715;N715;
N715c:=sub<N|N715>;#N715c;N715c;

/* Obtains the transversals of N(715).*/
T715:=Transversal(N,N715c);#T715;T715;

/*Obtains the total number of cosets(through [715] & excluding [*])
by conjugation. */
for i in [1..#T715] do
ss:=[7,15]^T715[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;
```

```
for g in N715c do g; end for;

/*Checks the orbits of N715c */
Orbits(N715c);

/*================
6.  [7, 28]
==============*/
S:={[7,28]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[28] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [728]. */
N728:=Stabiliser(N7,28);#N728;N728;
N728c:=sub<N|N728>;#N728c;N728c;

/* Obtains the transversals of N(728).*/
T728:=Transversal(N,N728c);#T728;T728;

/*Obtains the total number of cosets(through [728] & excluding [*])
by conjugation. */
for i in [1..#T728] do
ss:=[7,28]^T728[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

for g in N728c do g; end for;

/*Checks the orbits of N728c */
Orbits(N728c);

/*================
7.  [7, 14]
==============*/
S:={[7,14]};
SS:=S^N;
```

```
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[14] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [56]. */
N714:=Stabiliser(N7,14);#N714;N714;
N714c:=sub<N|N714>;#N714c;N714c;

/* Obtains the transversals of N(714).*/
T714:=Transversal(N,N714c);#T714;T714;

/*Obtains the total number of cosets(through [71] & excluding [*])
by conjugation. */
for i in [1..#T714] do
ss:=[7,14]^T714[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

for g in N714c do g; end for;

/*Checks the orbits of N714c */
Orbits(N714c);

/*================
8. [21,14]
==============*/
S:={[21,14]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[21]*ts[14] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [56]. */
N2114:=Stabiliser(N21,14);#N2114;N2114;
N2114c:=sub<N|N2114>;#N2114c;N2114c;
```

```
/* Obtains the transversals of N(2114).*/
T2114:=Transversal(N,N2114c);#T2114;T2114;

/*Obtains the total number of cosets(through [2114] & excluding [*])
by conjugation. */
for i in [1..#T2114] do
ss:=[21,14]^T2114[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

for g in N2114c do g; end for;

/*Checks the orbits of N2114c */
Orbits(N2114c);

/*================
9. [21,15]
===============*/
S:={[21,15]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[21]*ts[15] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [2115]. */
N2115:=Stabiliser(N21,15);#N2115;N2115;
N2115c:=sub<N|N2115>;#N2115c;N2115c;

/* Obtains the transversals of N(2115).*/
T2115:=Transversal(N,N2115c);#T2115;T2115;

/*Obtains the total number of cosets(through [2115] & excluding [*])
by conjugation. */
for i in [1..#T2115] do
ss:=[21,15]^T2115[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;
```

```
for g in N2115c do g; end for;

/*Checks the orbits of N2115c */
Orbits(N2115c);

/*================
10. [21,1]
==============*/
S:={[21,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[21]*ts[1] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [56]. */
N211:=Stabiliser(N21,1);#N211;N211;
N211c:=sub<N|N211>;#N211c;N211c;

/* Obtains the transversals of N(211).*/
T211:=Transversal(N,N211c);#T211;T211;

/*Obtains the total number of cosets(through [211] & excluding [*])
by conjugation. */
for i in [1..#T211] do
ss:=[21,1]^T211[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

for g in N211c do g; end for;

/*Checks the orbits of N211c */
Orbits(N211c);

/*================
11. [7,1,3]
==============*/
S:={[7,1,3]};
SS:=S^N;
```

```
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[3] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,3]. */
N713:=Stabiliser(N71,3);#N713;N713;
N713c:=sub<N|N713>;#N713c;N713c;

/* Obtains the transversals of N(713).*/
T713:=Transversal(N,N713c);#T713;T713;

/*Obtains the total number of cosets(through [713] & excluding [*])
by conjugation. */
for i in [1..#T713] do
ss:=[7,1,3]^T713[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N713 */
for g in N713c do g; end for;

/*Checks the orbits of N713c */
Orbits(N713c);

/*================
12. [7,1,4]
==============*/
S:={[7,1,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[4] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,4]. */
N714:=Stabiliser(N71,4);#N714;N714;
N714c:=sub<N|N714>;#N714c;N714c;
```

```
/* Obtains the transversals of N(714).*/
T714:=Transversal(N,N714c);#T714;T714;

/*Obtains the total number of cosets(through [714] & excluding [*])
by conjugation. */
for i in [1..#T714] do
ss:=[7,1,4]^T713[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N714 */
for g in N714c do g; end for;

/*Checks the orbits of N714c */
Orbits(N714c);

/*================
13. [7,1,6]
==============*/
S:={[7,1,6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[6] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,6]. */
N716:=Stabiliser(N71,6);#N716;N716;
N716c:=sub<N|N716>;#N716c;N716c;

/* Obtains the transversals of N(716).*/
T716:=Transversal(N,N716c);#T716;T716;

/*Obtains the total number of cosets(through [716] & excluding [*])
by conjugation. */
for i in [1..#T716] do
ss:=[7,1,6]^T716[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
```

```
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N716 */
for g in N716c do g; end for;

/*Checks the orbits of N716c */
Orbits(N716c);

/*================
14. [7,1,7]
==============*/
S:={[7,1,7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[7] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* obtain relation [14,6,14] ~ [7,1,7]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[1]*ts[7] eq n*(ts[14]*ts[6]*ts[14]) then print n;
end if; end for;

/* to check we do the following:*/
 ts[7]*ts[1]*ts[7] eq ((fy)^2)*(ts[14]*ts[6]*ts[14]);

/*Finds the coset stabilizer of [7,1,7]. */
N717:=Stabiliser(N71,7);#N717;N717;
N717c:=sub<N|N717>;#N717c;N717c;


/* Enters the relation  [14,6,14] ~ [7,1,7] */
for n in N do if 14^n eq 7 and 6^n eq 1
then N717c:=sub<N|N717c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,1,7]^N717c;
```

```
/* Obtains the transversals of N(717).*/
T717:=Transversal(N,N717c);#T717;T717;

/*Obtains the total number of cosets(through [7,1,7] & excluding [*])
 by conjugation. */
for i in [1..#T717] do
ss:=[7,1,7]^T717[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N717 */
for g in N717c do g; end for;

/*Checks the orbits of N717c */
Orbits(N717c);

/*================
15. [7,1,8]
==============*/
S:={[7,1,8]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[8] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,8]. */
N718:=Stabiliser(N71,8);#N718;N718;
N718c:=sub<N|N718>;#N718c;N718c;

/* Obtains the transversals of N(718).*/
T718:=Transversal(N,N718c);#T718;T718;

/*Obtains the total number of cosets(through [718] & excluding [*])
by conjugation. */
for i in [1..#T718] do
ss:=[7,1,8]^T718[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
```

```
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N718 */
for g in N718c do g; end for;

/*Checks the orbits of N718c */
Orbits(N718c);

/*================
16. [7,1,10]
==============*/
S:={[7,1,10]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[10] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* obtain relation [14,12,11] ~ [7,1,10]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[1]*ts[10] eq n*(ts[14]*ts[12]*ts[11]) then print n;
end if; end for;

/* to check we do the following:
ts[7]*ts[1]*ts[10] eq (f(n)*(ts[14]*ts[12]*ts[11]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,1,10]. */
N7110:=Stabiliser(N71,10);#N7110;N7110;
N7110c:=sub<N|N7110>;#N7110c;N7110c;

/* Enters the relation  [14,12,11] ~ [7,1,10] */
for n in N do if 14^n eq 7 and 12^n eq 1 and 11^n eq 10
then N7110c:=sub<N|N7110c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,1,10]^N7110c;

/* Obtains the transversals of N(7110).*/
```

```
T7110:=Transversal(N,N7110c);#T7110;T7110;

/*Obtains the total number of cosets(through [7110] & excluding [*])
by conjugation. */
for i in [1..#T7110] do
ss:=[7,1,10]^T7110[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7110 */
for g in N7110c do g; end for; #N7110c;

/*Checks the orbits of N7110c */
Orbits(N7110c);

/*================
17. [7,1,11]
==============*/
S:={[7,1,11]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[11] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,11]. */
N7111:=Stabiliser(N71,11);#N7111;N7111;
N7111c:=sub<N|N7111>;#N7111c;N7111c;

/* Obtains the transversals of N(7111).*/
T7111:=Transversal(N,N7111c);#T7111;T7111;

/*Obtains the total number of cosets(through [7111] & excluding [*])
by conjugation. */
for i in [1..#T7111] do
ss:=[7,1,11]^T7111[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;
```

```
/* prints elements in the stabilizer, N7111 */
for g in N7111c do g; end for;

/*Checks the orbits of N7111c */
Orbits(N7111c);

/*================
18. [7,1,12]
==============*/
S:={[7,1,12]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[12] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,12]. */
N7112:=Stabiliser(N71,12);#N7112;N7112;
N7112c:=sub<N|N7112>;#N7112c;N7112c;

/* Obtains the transversals of N(7112).*/
T7112:=Transversal(N,N7112c);#T7112;T7112;

/*Obtains the total number of cosets(through [7112] & excluding [*])
by conjugation. */
for i in [1..#T7112] do
ss:=[7,1,12]^T7112[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7112 */
for g in N7112c do g; end for;

/*Checks the orbits of N7112c */
Orbits(N7112c);

/*================
19. [7,1,17]
==============*/
```

```
S:={[7,1,17]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[17] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* obtain relation [5,9,24] ~ [7,1,17]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[1]*ts[17] eq n*(ts[5]*ts[9]*ts[24]) then print n;
end if; end for;

/* to check we do the following:
 ts[7]*ts[1]*ts[17] eq (f(n))*(ts[5]*ts[9]*ts[24]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,1,17]. */
N7117:=Stabiliser(N71,17);#N7117;N7117;
N7117c:=sub<N|N7117>;#N7117c;N7117c;

/* Enters the relation  [5,9,24] ~ [7,1,17]  */
for n in N do if 5^n eq 7 and 9^n eq 1 and 24^n eq 17
then N7117c:=sub<N|N7117c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,1,17]^N7117c;

/* Obtains the transversals of N(7117).*/
T7117:=Transversal(N,N7117c);#T7117;T7117;

/*Obtains the total number of cosets(through [7117] & excluding [*])
 by conjugation. */
for i in [1..#T7117] do
ss:=[7,1,17]^T7117[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;
```

```
/* prints elements in the stabilizer, N7117 */
for g in N7117c do g; end for;

/*Checks the orbits of N7117c */
Orbits(N7117c);

/*================
20. [7,1,18]
==============*/
S:={[7,1,18]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[18] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,18]. */
N7118:=Stabiliser(N71,18);#N7118;N7118;
N7118c:=sub<N|N7118>;#N7118c;N7118c;

/* Obtains the transversals of N(7118).*/
T7118:=Transversal(N,N7118c);#T7118;T7118;

/*Obtains the total number of cosets(through [7118] & excluding [*])
by conjugation. */
for i in [1..#T7118] do
ss:=[7,1,18]^T7118[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7118 */
for g in N7118c do g; end for;

/*Checks the orbits of N7118c */
Orbits(N7118c);

/*================
21. [7,1,19]
==============*/
S:={[7,1,19]};
```

```
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[19] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,19]. */
N7119:=Stabiliser(N71,19);#N7119;N7119;
N7119c:=sub<N|N7119>;#N7119c;N7119c;

/* Obtains the transversals of N(7119).*/
T7119:=Transversal(N,N7119c);#T7119;T7119;

/*Obtains the total number of cosets(through [7119] & excluding [*])
by conjugation. */
for i in [1..#T7119] do
ss:=[7,1,19]^T7119[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7119 */
for g in N7119c do g; end for;

/*Checks the orbits of N7119c */
Orbits(N7119c);

/*================
22. [7,1,20]
==============*/
S:={[7,1,20]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[20] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* obtains relation [7,1,20] ~ [14,2,24]
To enter relation, first we verify if it holds by finding n .
```

```
Then we enter relation. */

for n in IN do
if ts[7]*ts[1]*ts[20] eq n*(ts[14]*ts[2]*ts[24]) then print n;
end if; end for;

/* to check we do the following:
 ts[7]*ts[1]*ts[20] eq (f(n))*(ts[14]*ts[2]*ts[24]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,1,20]. */
N7120:=Stabiliser(N71,20);#N7120;N7120;
N7120c:=sub<N|N7120>;#N7120c;N7120c;

/* Enters the relation  [14,2,24] ~ [7,1,20]  */
for n in N do if 14^n eq 7 and 2^n eq 1 and 24^n eq 20
then N7120c:=sub<N|N7120c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,1,20]^N7120c;

/* Obtains the transversals of N(7120).*/
T7120:=Transversal(N,N7120c);#T7120;T7120;

/*Obtains the total number of cosets(through [7120] & excluding [*])
by conjugation. */
for i in [1..#T7120] do
ss:=[7,1,20]^T7120[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7120 */
for g in N7120c do g; end for;

/*Checks the orbits of N7120c */
Orbits(N7120c);

/*================
23. [7,1,21]
==============*/
S:={[7,1,21]};
SS:=S^N;
```

```
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[21] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* Obtains the relation [7,1,21] ~ [14,1,28]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[1]*ts[21] eq n*(ts[14]*ts[1]*ts[28]) then print n;
end if; end for;

/* to check we do the following:
 ts[7]*ts[1]*ts[21] eq (f(n))*(ts[14]*ts[1]*ts[28]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,1,21]. */
N7121:=Stabiliser(N71,21);#N7121;N7121;
N7121c:=sub<N|N7121>;#N7121c;N7121c;

/* Enters the relation  [14,1,28] ~ [7,1,21]  */
for n in N do if 14^n eq 7 and 1^n eq 1 and 28^n eq 21
then N7121c:=sub<N|N7121c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,1,21]^N7121c;

/* Obtains the transversals of N(7121).*/
T7121:=Transversal(N,N7121c);#T7121;T7121;

/*Obtains the total number of cosets(through [7121] & excluding [*])
by conjugation. */
for i in [1..#T7121] do
ss:=[7,1,21]^T7121[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7121 */
for g in N7121c do g; end for;
```

```
/*Checks the orbits of N7121c */
Orbits(N7121c);

/*================
24. [7,1,22]
===============*/
S:={[7,1,22]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[22] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,22]. */
N7122:=Stabiliser(N71,22);#N7122;N7122;
N7122c:=sub<N|N7122>;#N7122c;N7122c;

/* Obtains the transversals of N(7122).*/
T7122:=Transversal(N,N7122c);#T7122;T7122;

/*Obtains the total number of cosets(through [7122] & excluding [*])
by conjugation. */
for i in [1..#T7122] do
ss:=[7,1,22]^T7122[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7122 */
for g in N7122c do g; end for;

/*Checks the orbits of N7122c */
Orbits(N7122c);

/*================
25. [7,1,23]
===============*/
S:={[7,1,23]};
SS:=S^N;
SSS:=Setseq(SS);
```

```
for i in[1..#SS] do
for g in IN do if ts[7]*ts[1]*ts[23] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,1,23]. */
N7123:=Stabiliser(N71,23);#N7123;N7123;
N7123c:=sub<N|N7123>;#N7123c;N7123c;

/* Obtains the transversals of N(7123).*/
T7123:=Transversal(N,N7123c);#T7123;T7123;

/*Obtains the total number of cosets(through [7123] & excluding [*])
by conjugation. */
for i in [1..#T7123] do
ss:=[7,1,23]^T7123[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7123 */
for g in N7123c do g; end for;

/*Checks the orbits of N7123c */
Orbits(N7123c);

/*================
26. [7,15,3]
==============*/
S:={[7,15,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[15]*ts[3] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* Obtains the relation [7,15,3] ~ [13,22,5]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */
```

```
for n in IN do
if ts[7]*ts[15]*ts[3] eq n*(ts[13]*ts[22]*ts[5]) then print n;
end if; end for;

/* to check we do the following:
 ts[7]*ts[15]*ts[3] eq (f(n))*(ts[13]*ts[22]*ts[5]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,15,3]. */
N7153:=Stabiliser(N715,3);#N7153;N7153;
N7153c:=sub<N|N7153>;#N7153c;N7153c;

/* Enters the relation  [13,22,5] ~ [7,15,3]  */
for n in N do if 13^n eq 7 and 22^n eq 15 and 5^n eq 3
then N7153c:=sub<N|N7153c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,15,3]^N7153c;

/* Obtains the transversals of N(7153).*/
T7153:=Transversal(N,N7153c);#T7153;T7153;

/*Obtains the total number of cosets(through [7153] & excluding [*])
by conjugation. */
for i in [1..#T7153] do
ss:=[7,15,3]^T7153[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7153 */
for g in N7153c do g; end for;

/*Checks the orbits of N7153c */
Orbits(N7153c);

/*================
27. [7,15,7]
==============*/
S:={[7,15,7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
```

```
for g in IN do if ts[7]*ts[15]*ts[7] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* Obtains the relation [7,15,7] ~ [1,21,1]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[15]*ts[7] eq n*(ts[1]*ts[21]*ts[1]) then print n;
end if; end for;

/* to check we do the following:
 ts[7]*ts[15]*ts[7] eq (f(n))*(ts[1]*ts[21]*ts[1]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,15,7]. */
N7157:=Stabiliser(N715,7);#N7157;N7157;
N7157c:=sub<N|N7157>;#N7157c;N7157c;

/* Enters the relation  [1,21,1] ~ [7,15,7]  */
for n in N do if 1^n eq 7 and 21^n eq 15
then N7157c:=sub<N|N7157c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,15,7]^N7157c;

/* Obtains the transversals of N(7157).*/
T7157:=Transversal(N,N7157c);#T7157;T7157;

/*Obtains the total number of cosets(through [7157] & excluding [*])
by conjugation. */
for i in [1..#T7157] do
ss:=[7,15,7]^T7157[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7157 */
#N7157;
for g in N7157c do g; end for;
```

```
/*Checks the orbits of N7157c */
Orbits(N7157c);

/*================
28. [7,15,8]
==============*/
S:={[7,15,8]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[15]*ts[8] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* Obtains the relation [7,15,8] ~ [9,19,12]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[15]*ts[8] eq n*(ts[9]*ts[19]*ts[12]) then print n;
end if; end for;

/* to check we do the following:
 ts[7]*ts[15]*ts[8] eq (f(n))*(ts[9]*ts[19]*ts[12]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,15,8]. */
N7158:=Stabiliser(N715,8);#N7158;N7158;
N7158c:=sub<N|N7158>;#N7158c;N7158c;

/* Enters the relation  [9,19,12] ~ [7,15,8]  */
for n in N do if 9^n eq 7 and 19^n eq 15 and 12^n eq 8
then N7158c:=sub<N|N7158c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,15,8]^N7158c;

/* Obtains the transversals of N(7158).*/
T7158:=Transversal(N,N7158c);#T7158;T7158;

/*Obtains the total number of cosets(through [7158] & excluding [*])
by conjugation. */
for i in [1..#T7158] do
```

```
ss:=[7,15,8]^T7158[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N71578*/
for g in N7158c do g; end for;

/*Checks the orbits of N7158c */
Orbits(N7158c);

/*================
29. [7,15,11]
===============*/
S:={[7,15,11]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[15]*ts[11] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,15,11]. */
N71511:=Stabiliser(N715,11);#N71511;N71511;
N71511c:=sub<N|N71511>;#N71511c;N71511c;

/* Obtains the transversals of N(71511).*/
T71511:=Transversal(N,N71511c);#T71511;T71511;

/*Obtains the total number of cosets(through [71511] & excluding [*])
by conjugation. */
for i in [1..#T71511] do
ss:=[7,15,11]^T71511[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N71511*/
for g in N71511c do g; end for;

/*Checks the orbits of N71511c */
```

```
Orbits(N71511c);

/*================
30.  [7,15,18]
===============*/
S:={[7,15,18]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[15]*ts[18] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* Obtains the relation [7,15,18] ~ [4,27,21]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[15]*ts[18] eq n*(ts[4]*ts[27]*ts[21]) then print n;
end if; end for;

/* to check we do the following:
 ts[7]*ts[15]*ts[18] eq (f(n))*(ts[4]*ts[27]*ts[21]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,15,18]. */
N71518:=Stabiliser(N715,18);#N71518;N71518;
N71518c:=sub<N|N71518>;#N71518c;N71518c;

/* Enters the relation  [4,27,21] ~ [7,15,18]  */
for n in N do if 4^n eq 7 and 27^n eq 15 and 21^n eq 18
then N71518c:=sub<N|N71518c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,15,18]^N71518c;

/* Obtains the transversals of N(71518).*/
T71518:=Transversal(N,N71518c);#T71518;T71518;

/*Obtains the total number of cosets(through [71518] & excluding [*])
by conjugation. */
for i in [1..#T71518] do
ss:=[7,15,18]^T71518[i];
```

```
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N71518*/
for g in N71518c do g; end for;

/*Checks the orbits of N71518c */
Orbits(N71518c);

/*================
31.  [7,15,20]
==============*/
S:={[7,15,20]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[15]*ts[20] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,15,20]. */
N71520:=Stabiliser(N715,20);#N71520;N71520;
N71520c:=sub<N|N71520>;#N71520c;N71520c;

/* Obtains the transversals of N(71520).*/
T71520:=Transversal(N,N71520c);#T71520;T71520;

/*Obtains the total number of cosets(through [71520] & excluding [*])
by conjugation. */
for i in [1..#T71520] do
ss:=[7,15,20]^T71520[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/*Checks the orbits of N71520c */
Orbits(N71520c);

/*================
32.  [7,15,21]
```

```
==============*/
S:={[7,15,21]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[15]*ts[21] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* Obtains the relation [7,15,21] ~ [13,22,27]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[15]*ts[21] eq n*(ts[13]*ts[22]*ts[27]) then print n;
end if; end for;

/* to check we do the following:
 ts[7]*ts[15]*ts[21] eq (f(n))*(ts[13]*ts[22]*ts[27]);
but i cannot calculate n in terms of x and y. */

/*Finds the coset stabilizer of [7,15,21]. */
N71521:=Stabiliser(N715,21);#N71521;N71521;
N71521c:=sub<N|N71521>;#N71521c;N71521c;

/* Enters the relation  [13,22,27] ~ [7,15,21]  */
for n in N do if 13^n eq 7 and 22^n eq 15 and 27^n eq 21
then N71521c:=sub<N|N71521c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,15,21]^N71521c;

/* Obtains the transversals of N(71521).*/
T71521:=Transversal(N,N71521c);#T71521;T71521;

/*Obtains the total number of cosets(through [71521] & excluding [*])
by conjugation. */
for i in [1..#T71521] do
ss:=[7,15,21]^T71521[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;
```

```
/* prints elements in the stabilizer, N71521*/
for g in N71521c do g; end for;

/*Checks the orbits of N71521c */
Orbits(N71521c);

/*================
33.  [7,28,7]
==============*/
S:={[7,28,7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[28]*ts[7] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/* Obtains the relation [7,28,7] ~ [14,21,14]
To enter relation, first we verify if it holds by finding n .
Then we enter relation. */

for n in IN do
if ts[7]*ts[28]*ts[7] eq n*(ts[14]*ts[21]*ts[14]) then print n;
end if; end for;

/* to check we do the following:*/
 ts[7]*ts[28]*ts[7] eq (fy^2)*(ts[14]*ts[21]*ts[14]);

/*Finds the coset stabilizer of [7,28,7]. */
N7287:=Stabiliser(N728,7);#N7287;N7287;
N7287c:=sub<N|N7287>;#N7287c;N7287c;

/* Enters the relation   [7,28,7] ~ [14,21,14]  */
for n in N do if 14^n eq 7 and 21^n eq 28
then N7287c:=sub<N|N7287c,n>; end if; end for;

/* To double check, the following should print all relations*/
[7,28,7]^N7287c;

/* Obtains the transversals of N(7287).*/
T7287:=Transversal(N,N7287c);#T7287;T7287;
```

```
/*Obtains the total number of cosets(through [71521] & excluding [*])
by conjugation. */
for i in [1..#T7287] do
ss:=[7,28,7]^T7287[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/* prints elements in the stabilizer, N7287*/
for g in N7287c do g; end for;

/*Checks the orbits of N7287c */
Orbits(N7287c);

/*================
34.  [7,28,15]
==============*/
S:={[7,28,15]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[28]*ts[15] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,28,15]. */
N72815:=Stabiliser(N728,15);#N72815;N72815;
N72815c:=sub<N|N72815>;#N72815c;N72815c;

/* Obtains the transversals of N(72815).*/
T72815:=Transversal(N,N72815c);#T72815;T72815;

/*Obtains the total number of cosets(through [72815] & excluding [*])
by conjugation. */
for i in [1..#T72815] do
ss:=[7,28,15]^T72815[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;
/*m= 3773*/
```

```
/*Checks the orbits of N72815c */
Orbits(N72815c);

/*================
35.  [7,14,15]
==============*/
S:={[7,14,15]};
SS:=S^N;
SSS:=Setseq(SS);
for i in[1..#SS] do
for g in IN do if ts[7]*ts[14]*ts[15] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

/*Finds the coset stabilizer of [7,14,15]. */
N71415:=Stabiliser(N714,15);#N71415;N71415;
N71415c:=sub<N|N71415>;#N71415c;N71415c;

/* Obtains the transversals of N(71415).*/
T71415:=Transversal(N,N71415c);#T71415;T71415;

/*Obtains the total number of cosets(through [71415] & excluding [*])
by conjugation. */
for i in [1..#T71415] do
ss:=[7,14,15]^T71415[i];
cst[prodim(1, ts, ss)]:= ss;
end for;
m:=0;
for i in [1..5280] do if cst[i] ne [] then m:=m+1; end if; end for; m;

/*Checks the orbits of N71415c */
Orbits(N71415c);
```

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